ON TWO GEOMETRIC REALIZATIONS OF AN AFFINE HECKE ALGEBRA

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Abstract. The article is a contribution to the local theory of geometric Langlands correspondence. The main result is a categorification of the isomorphism between the (extended) affine Hecke algebra associated to a semi-simple group G and Grothendieck group of equivariant coherent sheaves on Steinberg variety of Langlands dual group $G\check{}$; this isomorphism due to Kazhdan–Lusztig and Ginzburg is a key step in the proof of tamely ramified local Langlands conjectures.

The paper is a continuation of [1], [8], it relies on technical material developed in [13].

Contents

1. Introduction and statement of the result.

1.1. **Affine Hecke algebra and its two categorifications.** Let k be a field, and let $F = k((t)) \supset O = k[[t]]$ be the field of functions on the punctured formal disc over k and its ring of integers. Let G be a split semi-simple linear algebraic group over k; let $B \subset G$ be a Borel subgroup, and $I \subset G(F)$ be the corresponding Iwahori subgroup (thus I is the preimage of B under the evaluation map $G(O) \to G$).

If k is finite then the group $G(F)$ is a locally compact topological group, I is its open compact subgroup, and the space H of $\mathbb{C}$-valued finitely supported functions on the two-sided quotient $I\backslash G(F)/I$ carries an algebra structure under convolution; this is the Iwahori-Matsumoto Hecke algebra. Also $H = \mathcal{H} \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{C}$ where $\mathcal{H}$ is the (extended) affine Hecke algebra and the homomorphism $\mathbb{Z}[q^{\pm 1}] \to \mathbb{C}$ sends q to $|k|$.

Based on Grothendieck "sheaf-function" correspondence principle, one can consider the category of l-adic complexes (or perverse sheaves) on an $\mathbb{F}_q$-scheme (or on its base change to an algebraically closed field) as the categorical counterpart, or categorification, of the space of functions on the set of $\mathbb{F}_q$-points of the scheme; in particular, the space of functions is a quotient of the Grothedieck group of the category. This approach yields a certain derived category of etale sheaves which should be viewed as a categorification of the affine Hecke algebra $\mathcal{H}$.

On the other hand, as was discovered by Kazhdan and Lusztig (and independently by Ginzburg), the affine Hecke algebra can be realized as the Grothendieck group of equivariant coherent sheaves on the Steinberg variety of the Langlands dual group, thus the corresponding derived category of coherent sheaves provides another categorification of $\mathcal{H}$.

The goal of the present paper is to construct an equivalence between the two triangulated categories which categorify the affine Hecke algebra. A step in this direction has been made in the previous works [1], [8], where a geometric theory of the anti-spherical (Whittaker) module over H was developed; in the present paper we extend this analysis to the affine Hecke algebra itself.

The possibility to realize the affine Hecke algebra $\mathcal{H}$ and the "anti-spherical" module over it as Grothendieck groups of (equivariant) coherent sheaves on varieties related to $G\check{}$ plays a key role in the proof of classification of irreducible representations of H, which constitutes a particular case of local Langlands conjecture; see [21], and exposition in [14]. Thus one may hope that the categorification of these realizations proposed here can contribute to the geometric Langlands program. In fact, since the result of the paper was announced, it has been applied and generalized by several authors working in that area, see [16], [6], [10]. Let us point out that existence of (some variant of) such a categorification was proposed as a conjecture by V. Ginzburg (see Introduction to [14]).

1.2. **Statement of the result.** Let us now describe our result in more detail.

1.2.1. *Categories of l-adic sheaves.* Recall the well known group schemes $\mathbf{G_O} \supset \mathbf{I} \supset \mathbf{I}^0$ over k (of infinite type) such that $\mathbf{G_O}(k) = G(O)$, $\mathbf{I}(k) = I$, $\mathbf{I}^0(k) = I^0$ where I^0 is the pro-p radical of I; and a group ind-scheme $\mathbf{G_F}$ with $\mathbf{G_F}(k) = G(F)$. We also have the quotient ind-varieties: the affine Grassmanian $\mathcal{G}\mathfrak{r}$, the affine flag variety $\mathcal{F}\ell = \mathbf{G_F}/\mathbf{I}$ and the extended affine flag variety $\widetilde{\mathcal{F}\ell} = \mathbf{G_F}/\mathbf{I}^0$, see e.g. [17], Appendix, §A.5. Thus $\mathcal{G}\mathfrak{r}$, $\mathcal{F}\ell$, $\widetilde{\mathcal{F}\ell}$ are direct limits of finite dimensional varieties with transition maps being closed embeddings, in the case of $\mathcal{G}\mathfrak{r}$ and $\mathcal{F}\ell$ all the finite dimensional varieties in the direct system are projective. We have $\mathcal{G}\mathfrak{r}(k) = G(F)/G(O)$, $\mathcal{F}\ell(k) = G(F)/I$, $\widetilde{\mathcal{F}\ell}(k) = G(F)/I^0(k)$.

Let $D(\widetilde{\mathcal{F}\ell})$, $D(\mathcal{F}\ell)$, $D(\mathcal{G}\mathfrak{r})$ be the constructible derived categories of l-adic sheaves ($l \neq char(k)$; see [15], 1.1.2; [4] 2.2.14–2.2.18; and [17], §A.2 for (straightforward) generalization of the definition of an l-adic complex to a certain class of ind-schemes) on the respective spaces.

The protagonists of this paper are as follows. Let $D_{I,I} = D_I(\mathcal{F}\ell)$ be the $\mathbf{I}$-equivariant derived category of l-adic sheaves on $\mathcal{F}\ell$; $D_{I^0,I} = D_{I^0}(\mathcal{F}\ell)$ be the $\mathbf{I}^0$-equivariant derived category of l-adic sheaves on $\mathcal{F}\ell$, and let D_{I^0,I^0} be the full subcategory in the $\mathbf{I}^0$ equivariant derived category of $\widetilde{\mathcal{F}\ell}$ consisting of complexes whose cohomology is monodromic with respect to the right $\mathbf{T} = \mathbf{I}/\mathbf{I}^0$ action with unipotent monodromy.

The categories $D_{I,I}$ and D_{I^0,I^0} are equipped with an associative product operation provided by convolution; $D_{I,I}$ is unital while D_{I^0,I^0} lacks the unit object.[1] We

[1]Notice that convolution with an object of D_{I^0,I^0} involves direct image under a non-proper morphism, thus convolution could be defined in two different ways, using either direct image or direct image with compact support. We use the version with the ordinary direct image.

have commuting actions of D_{I^0,I^0} and $D_{I,I}$ on $D_{I^0,I}$ by left and right convolution respectively. The convolution operation will be denoted by $*$.

Let $\mathcal{P}_{I,I} \subset D_{I,I}$, $\mathcal{P}_{I^0,I} \subset D_{I^0,I}$, $\mathcal{P}_{I^0,I^0} \subset D_{I^0,I^0}$ be the subcategories of perverse sheaves. A standard argument shows that

$$D^b(\mathcal{P}_{I^0,I}) \cong D_{I^0,I} \qquad (1)$$
$$D^b(\mathcal{P}_{I^0,I^0}) \cong D_{I^0,I^0},$$

while the natural functor $D^b(\mathcal{P}_{I,I}) \to D_{I,I}$ is not an equivalence.

1.2.2. *The dual side.* Let $G\check{}$ be the Langlands dual group over the field $\overline{\mathbb{Q}_l}$. The goal of the paper is to provide a description for the above categories in terms of $G\check{}$. To formulate the answer we need to recall the following construction.

Let $X \to Y$, $X' \to Y$ be morphisms of algebraic varieties. To simplify the statements we will assume that X, X', Y are varieties over a field k, Y is smooth and morphisms $X \to Y$, $X' \to Y$ are proper.

One can consider the *derived fiber product* $X \overset{\mathrm{L}}{\times}_Y X'$ which is a *differential graded scheme* (DG-scheme for short), and the triangulated category $DGCoh(X \overset{\mathrm{L}}{\times}_Y X')$.

If $Tor_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_{X'}) = 0$ for $i > 0$ then the derived fiber product reduces to the ordinary fiber product and $DGCoh(X \overset{\mathrm{L}}{\times}_Y X') = D^b(Coh(X \times_Y X'))$.

The triangulated category $DGCoh(X \overset{\mathrm{L}}{\times}_Y X)$ has a natural monoidal structure provided by convolution. The category $D^b(Coh(X))$ is naturally a module category for the monoidal category $DGCoh(X \overset{\mathrm{L}}{\times}_Y X)$. [For example, when X is a finite set and Y is a point the induced structures on the Grothendieck group amounts to matrix multiplication and the action of $n \times n$ matrices on n-vectors respectively]. More generally, the category $DGCoh(X \overset{\mathrm{L}}{\times}_Y X')$ has two commuting actions: the action of $DGCoh(X \overset{\mathrm{L}}{\times}_Y X)$ on the left and an action of $DGCoh(X' \overset{\mathrm{L}}{\times}_Y X')$ on the right.

Given an action of an affine algebraic group H on X, X', Y compatible with the maps, one gets equivariant versions of the above statements.

We will apply this in the following situation. We let $Y = \mathfrak{g}\check{}$ be the Lie algebra of $G\check{}$, $X = \tilde{\mathfrak{g}}\check{} = \{(\mathfrak{b}, x) \mid \mathfrak{b} \in \mathcal{B},\, x \in \mathfrak{b}\}$, $X' = \tilde{\mathcal{N}} = \{(\mathfrak{b}, x) \mid \mathfrak{b} \in \mathcal{B},\, x \in rad(\mathfrak{b})\}$, where $\mathcal{B}$ is the flag variety for $G\check{}$ parametrizing Borel subalgebras in $\mathfrak{g}\check{}$.

A standard complete intersection argument shows that the Tor vanishing condition holds for the fiber products $\tilde{\mathfrak{g}}\check{} \times_{\mathfrak{g}\check{}} \tilde{\mathfrak{g}}\check{}$ and $\tilde{\mathfrak{g}}\check{} \times_{\mathfrak{g}\check{}} \tilde{\mathcal{N}}$; however, it does not hold for the product $\tilde{\mathcal{N}} \times_{\mathfrak{g}\check{}} \tilde{\mathcal{N}}$.

We set $St = \tilde{\mathfrak{g}}\check{} \times_{\mathfrak{g}\check{}} \tilde{\mathfrak{g}}\check{}$, $St' = \tilde{\mathfrak{g}}\check{} \times_{\mathfrak{g}\check{}} \tilde{\mathcal{N}}$.

1.2.3. *Statement of the result.* We now formulate the main result of the paper.

For an algebraic variety X and a closed subset $Z \subset X$ we will let $Coh_Z(X)$ denote the full subcategory in $Coh(X)$ consisting of sheaves set-theoretically supported on Z. For a map $f : X \to Y$ and a closed subset $Z \subset Y$ we will abbreviate $Coh_{f^{-1}(Z)}(X)$ to $Coh_Z(X)$.

Theorem 1. *There exist natural equivalences of categories:*

$$\Phi_{I_0,I_0} : D_{I^0,I^0} \cong D^b(Coh^{G\check{}}_{\mathcal{N}}(St)), \qquad (2)$$

$$\Phi_{I_0,I} : D_{I^0,I} \cong D^b(Coh^{G\check{}}(St')), \qquad (3)$$

$$\Phi_{I,I}: D_{I,I} \cong DGCoh^{G\check{}}(\tilde{\mathcal{N}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}\check{}} \tilde{\mathcal{N}})). \tag{4}$$

Equivalences (2) *and* (4) *are compatible with the convolution product, while* (3) *is compatible with the action of the categories from* (2) *and* (4).

1.3. **The action on the Iwahori-Whittaker category.** It was pointed out above that the monoidal category of DG coherent (equivariant) sheaves on a fiber product $X \overset{\mathrm{L}}{\times}_Y X$ admits a natural action on the derived category of (equivariant) coherent sheaves on X. In particular, monoidal category $DGCoh^{G\check{}}(\tilde{\mathcal{N}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}\check{}} \tilde{\mathcal{N}})$ acts on $D^bCoh(\tilde{\mathcal{N}})$, while $D^bCoh^{G\check{}}(St)$ acts on $D^bCoh^{G\check{}}(\mathfrak{g}\check{})$.

To describe the corresponding structures on the loop group side, recall the category of *Iwahori-Whittaker* sheaves. The quotient $\mathbf{I}^0$ by its commutant is the sum of copies of the additive group indexed by vertices of the affine Dynkin graph. Fix an additive character ψ of $\mathbf{I}^0$ which is trivial on the summand of $\mathbf{I}^0/(\mathbf{I}^0)'$ corresponding to the affine root(s) and is non zero on the other summands. We denote by D^I_{IW} the $\mathbf{I}$ equivariant derived category of l-adic sheaves on the principal homogeneous space $\mathbf{G_F}/\mathbf{I}'_0$ which satisfies the ψ-equivariance condition with respect to the right action of $\mathbf{I}^0/\mathbf{I}'_0$, see [1].[2] We let $D^{I^0}_{IW}$ denote the category of $\mathbf{I}$ monodromic sheaves with unipotent monodromy on $\mathbf{G_F}/\mathbf{I}'_0$ which are ψ-equivariant with respect to the right action of $\mathbf{I}^0/\mathbf{I}'_0$, this is a particular case of the category considered in [13] (again, one needs to switch left with right to get from the present setting to that of [13].

The categories D_{I^0,I^0}, $D_{I,I}$ act on $D^{I^0}_{IW}$, D^I_{IW} respectively by convolution.

Theorem 2. *There exist equivalences of categories*

$$\Phi^I_{IW}: D^b(Coh^{G\check{}}(\tilde{\mathcal{N}})) \widetilde{\longrightarrow} D^I_{IW}, \tag{5}$$

$$\Phi^{I^0}_{IW}: D^b(Coh^{G\check{}}_{\tilde{\mathcal{N}}}(\tilde{\mathfrak{g}}\check{}))) \widetilde{\longrightarrow} D^{I^0}_{IW}, \tag{6}$$

satisfying the following compatibilities:

The equivalence $\Phi^{I^0}_{IW}$ is compatible with the action of $D^bCoh^{G\check{}}_{\mathcal{N}}(St)$ coming from the action of D_{I^0,I^0} on $D^{I^0}_{IW}$ and equivalence (2).

The equivalence Φ^I_{IW} is compatible with the action of $DGCoh^{G\check{}}(\tilde{\mathcal{N}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}\check{}} \tilde{\mathcal{N}})$ coming from the action of $D_{I,I}$ on D^I_{IW} and equivalence (4).

The equivalence (5) has been established in [1], and (6) can obtained by a similar argument, see below.

1.4. **Our strategy from the Hecke algebra point of view.** Some of the constructions exploited here are sheaf-theoretic analogs of known results in the theory of affine Hecke algebras.

Recall that $\mathcal{H}$ has a standard basis T_w indexed by elements w in the extended affine Weyl group W.

Let Λ be the coweight lattice of G and $\Lambda^+ \subset \Lambda$ be set of dominant weights. There exists a unique system of elements $\theta_\lambda \in \mathcal{H}$, $\lambda \in \Lambda$, such that $\theta_\lambda\theta_\mu = \theta_{\lambda+\mu}$ for all λ, $\mu \in \Lambda$ and $\theta_\lambda = T_\lambda$ for $\lambda \in \Lambda^+$. The categorification of the elements θ_λ are the so-called *Wakimoto sheaves*, see [1] and section 3.3 below.

[2] For the sake of convenience the conventions here differ from those of [1] by switching the roles of left and right multiplication. The resulting categories are easily seen to be equivalent.

The elements θ_λ span a commutative subalgebra $A \subset \mathcal{H}$ which contains the center $Z(\mathcal{H})$ of the affine Hecke algebra. Categorification of the center is provided by the work of Gaitsgory [17]. Categorification of the formula expressing central elements as linear combinations of θ_λ is the fact that central sheaves of [17] admit a filtration whose associated graded is a sum of Wakimoto sheaves, see [1] and section 3.5 below. This filtration plays a key role in our construction, see [1] and section 4.2, yielding a categorification of the isomorphism

$$A \cong K^0(Coh^{G\check{}}(\tilde{\mathfrak{g}}\check{})) \xrightarrow{\delta_*} K^0(Coh^{G\check{}}(St)), \tag{7}$$

where $\delta : \tilde{\mathfrak{g}}\check{} \to St$ is the diagonal embedding.

Another ingredient important to us is the *q-analog of the Schur anti-symmetrizer, or anti-spherical projector* $\xi = \sum\limits_{w \in W_f} (-1)^{\ell(w)} T_w$. Its relevance to representation theory of p-adic groups comes from the fact that the left ideal $\mathcal{H}\xi$ is canonically isomorphic to I invariants in the space of *Whittaker functions* on G.

The categorical counterpart of ξ is the maximal projective object in the category of perverse sheaves $G/B \cong \mathbf{G_O}/\mathbf{I}$ equivariant with respect to $\mathbf{I}^0$, it is discussed in section 5. Under the equivalence with coherent sheaves category that object corresponds to the structure sheaf of Steinberg variety.

Let $\mathcal{H}_{perf} \subset \mathcal{H}$ be the two-sided ideal generated by ξ. The full subcategory $D^{G\check{}}_{perf}(St) \subset D^b(Coh^{G\check{}}(St))$ of perfect complexes can be considered as a categorification of $\mathcal{H}_{perf}$. Furthermore, it is easy to see that $\mathcal{H}_{perf}$ is freely generated by ξ as a module over $A \otimes_Z A$. This allows one to deduce an equivalence between the two categorifications of H_{perf} from the categorification of (7). The subcategory $D^{G\check{}}_{perf}(St)$ is dense in $D^b(Coh^{G\check{}}(St))$ in an appropriate sense, which allows to extend the equivalence from the subcategory to the whole category.

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In this text we follow the original plan conceived more than a decade ago and treat the issues of homological algebra by ad hoc methods, using explicit DG models for triangulated categories of constructible sheaves based on generalized tilting sheaves. While the properties of tilting sheaves established in the course of the argument are (in the author's opinion) of an independent interest, it is likely that recent advances in homotopy algebra can be used to develop an alternative approach.

2. Outline of the argument

2.1. **Further notations and conventions.** We let $B \supset N$ be a Borel subgroup and its radical, and $N\check{} \subset B\check{}$ be similar subgroups in $G\check{}$.

We let Λ be the coweight lattice of G (identified with the weight lattice of $G\check{}$), W_f will denote the Weyl group and $W = W_f \ltimes \Lambda$ the extended affine Weyl group; $\ell : W \to \mathbb{Z}_{\geq 0}$ is the length function, and $\Lambda^+ \subset \Lambda$ is the set of dominant coweights, $w_0 \in W_f$ is the longest element.

We let $W^f \subset W$ be the subset of minimal length representatives of right cosets W/W_f. Notice that $\Lambda^+ \subset W^f$.

For $\lambda \in \Lambda$ we let $\mathcal{O}_{\mathcal{B}}(\lambda)$ be the corresponding line bundle on $\mathcal{B} = G\check{}/B\check{}$. We normalize the bijection between weights and line bundles so that Λ^+ corresponds to semi-ample line bundles. We let V_λ be an irreducible $G\check{}$-module such that $V_\lambda \otimes \mathcal{O}_{\mathcal{B}}$ maps to $\mathcal{O}_{\mathcal{B}}(\lambda)$ (so it might be called the representation with highest weight $-w_0(\lambda)$).

The ind-schemes $\mathcal{F}\ell = \mathbf{G_F}/\mathbf{I}$, $\widetilde{\mathcal{F}\ell} = \mathbf{G_F}/\mathbf{I}^0$ were introduced above, and the categories $D_{II} \supset \mathcal{P}_{II}$, $D_{I^0I} \supset \mathcal{P}_{I^0I}$, $D_{I^0I^0} \supset \mathcal{P}_{I^0I}$ were introduced above. We abbreviate $\mathcal{P} = \mathcal{P}_{I^0I}$ and let $\hat{\mathcal{P}}$, $\hat{D}$ be the pro-completions of $D_{I^0I^0}$ and $\mathcal{P}_{I^0I^0}$ respectively, see section 3.

Let $\pi : \widetilde{\mathcal{F}\ell} \to \mathcal{F}\ell$ be the projection.

The $\mathbf{I}$ orbits on $\mathcal{F}\ell$ are indexed by W, for $w \in W$ we let $j_w : \mathcal{F}\ell_w \to \mathcal{F}\ell$ be the embedding of the corresponding orbit. We have $\dim(\mathcal{F}\ell_w) = \ell(w)$.

We have standard objects $j_{w!} := j_{w!}(\underline{\overline{\mathbb{Q}_l}}[\ell(w)])$ and costandard object $j_{w*} = j_{w*}(\underline{\overline{\mathbb{Q}_l}}[\ell(w)])$ in $\mathcal{P}$. Their counterparts in $\hat{D}$ are the free monodromic (co)standard objects ∇_w, Δ_w, see sections 3.1, 3.2.

We also consider the Iwahori-Whittaker categories $D^I_{IW} \supset \mathcal{P}^I_{IW}$, $D^{I^0}_{IW} \supset \mathcal{P}^{I^0}_{IW}$, the pro-completions $\hat{D}_{IW}$, $\hat{\mathcal{P}}_{IW}$ of, respectively, $D^{I^0}_{IW}$, $\mathcal{P}^{I^0}_{IW}$, (co)standard objects $j^{IW}_{w!}, j^{IW}_{w*} \in \mathcal{P}^I_{IW}$ and free monodromic (co)standard object $\Delta_{w,IW}$, $\nabla_{w,IW} \in \hat{\mathcal{P}}_{IW}$, $w \in W^f$.

Recall that $St = \tilde{\mathfrak{g}}\check{} \times_{\mathfrak{g}\check{}} \tilde{\mathfrak{g}}\check{}$, let $p_{Spr,1} : St \to \tilde{\mathfrak{g}}\check{}$, $p_{Spr,2} : St \to \tilde{\mathfrak{g}}\check{}$ be the two projections. Also $St' = \tilde{\mathfrak{g}}\check{} \times_{\mathfrak{g}\check{}} \tilde{\mathcal{N}}$ with two projections $p'_{Spr,1} : St' \to \tilde{\mathfrak{g}}\check{}$, $p'_{Spr,2} : St' \to \tilde{\mathcal{N}}$. Let $\widehat{St} = \widehat{\mathfrak{g}}\check{} \times_{\mathfrak{g}\check{}} St$, where $\widehat{\mathfrak{g}}\check{}$ is the spectrum[3] of completion of the ring of functions $\mathcal{O}(\mathfrak{g}\check{})$ at the ideal of the point 0.

For an algebraic group H acting on an (ind)-scheme X we let $D_H(X)$ denote the equivariant derived category of H-equivariant constructible sheaves on X and let $Perv_H(X) \subset D_H(X)$ be the subcategory of perverse sheaves. Given a subgroup $K \subset H$ we have the functor of restricting the equivariance $Res^H_K : D_H(X) \to D_K(H)$ and the left adjoint functor $Av^H_K : D_K(X) \to D_H(X)$ (the latter can be thought of as the direct image for the morphism of stacks $X/K \to X/H$).

In particular, we have functors $Av^I_{I^0} : D_{I^0I} \to D_{II}$, $Av_{IW} : D_{I^0I^0} \to D^{I^0}_{IW}$, $Av^{I^0}_{IW} : D_{I^0I^0} \to D^{I^0}_{IW}$ etc. Notice that the first functor involves the left action of I^0, while the latter two have to do with the right action; when the action used may not be clear from the context we use notation $Av^{left}_{I^0}$, $Av^{right}_{I^0}$ to distinguish between the two.

2.2. **Idea of the argument: structural aspects.** The functor from the coherent category to the constructible one stems from certain natural structures on the constructible category. To describe the mechanism of obtaining such a functor from the additional structures on the target category it is convenient to use the concept of a triangulated category $\mathcal{C}$ *over* a stack X.

2.2.1. *Linear structure over a stack.* We refer to [19] for the notion of an abelian category over an algebraic stack, and to the forthcoming work [20] for the generalization to triangulated (or rather homotopy theoretic) context. For our present

[3]Alternatively we could work with completion defined as a formal scheme, the resulting category of coherent sheaves would be equivalent.

purposes it suffices to use the following simplified version of this concept. Let S be an algebraic stack and $\mathcal{C}$ a triangulated category (in all our example $S = X/G$ where X is a quasi-projective algebraic variety and G is a reductive algebraic group). The subcategory of perfect complexes $D_{perf}(S) \subset D^b(Coh(S))$ is a triangulated tensor category under the usual tensor product of coherent sheaves. By an *S-linear structure* on $\mathcal{C}$ we will mean an action of the tensor category $D_{perf}(S)$ on $\mathcal{C}$ compatible with the triangulated structure.

We now list basic classes of examples of such a structure to be used below.

(1) If $S = Spec(R)$ is an affine scheme, then for an R-linear abelian category $\mathcal{A}$ the triangulated category $D^b(\mathcal{A})$ acquires a natural S-linear structure.

(2) Let $S = pt/H$ where H is a linear algebraic group. If an abelian category $\mathcal{A}$ is a module category for the tensor category $Rep(H)$ of algebraic (finite dimensional) representations acting by exact functors, then $D^b(\mathcal{A})$ is an S-linear triangulated category.

(3) Combining the first two examples, assume now that $S = Spec(R)/H$ is a quotient of an affine scheme by a linear algebraic group action. Let $\mathcal{A}$ be an abelian category which is a module category for $Rep(H)$ acting on $\mathcal{A}$ by exact functors. Then we can define a new (in general not abelian) *"deequivariantized"* category $\mathcal{A}_{deeq}$ by setting $Ob(\mathcal{A}_{deeq}) = Ob(\mathcal{A})$, $Hom_{Ind(\mathcal{A})}(X, \underline{\mathcal{O}}_H(Y))$ where $Ind(\mathcal{A})$ stands for the category of Ind-objects in $\mathcal{A}$ and $\underline{\mathcal{O}}_H \in Ind(Rep(H))$ denotes the space of regular functions on H with H acting by left translations (thus $\underline{\mathcal{O}}_H$ can be considered as an ind-object in $Rep(H)$), see section 4.2.1 below for further details.

    Then $\mathcal{A}_{deeq}$ is a category enriched over the category of algebraic (not necessarily finite dimensional) representations of H. Then it is easy to see that an R-linear structure on $\mathcal{A}_{deeq}$ which is compatible with H-action induces an S-linear structure on $D^b(\mathcal{A})$.

(4) Suppose we are given an open embedding of algebraic stacks $S \to S'$ and a category $\mathcal{C}$ with an S'-linear structure. It is known that $D_{perf}(S)$ is the Karoubi (idempotent) completion of the quotient $D_{perf}(S')/D_{perf}(S')_{\partial S'}$, where $D_{perf}(S')_{\partial S'}$ is the full subcategory of perfect complexes on S' whose restriction to S vanishes. Thus if $\mathcal{C}$ is a Karoubian (idempotent complete) category, then an S'-linear structure on $\mathcal{C}$ such that $D_{perf}(S')_{\partial S'}$ acts by zero induces an S-linear structure on $\mathcal{C}$.

(5) One can use a variant of Serre's description of the category of coherent sheaves on a projective variety as a quotient of the category of graded modules over the homogeneous coordinate ring to devise a procedure for constructing an S linear structure for more general stacks S.

    Suppose that $S = X/H$ where X is a quasi-projective variety with an action of an affine algebraic group H. Assume that a linearization of the action, i.e. a linear action of H on a projective space $\mathbb{P}^n$ together with an equivariant locally closed embedding $X \to \mathbb{P}^n$, is fixed. Let $C \subset \mathbb{A}^{n+1}$ be the cone over the closure $\overline{X}$ of X in $\mathbb{P}^n$. Then C is an affine variety acted upon by $H \times \mathbb{G}_m$ and we have an open embedding $S \to S' = C/(H \times \mathbb{G}_m)$. Thus an S linear structure on $\mathcal{C} = D^b(\mathcal{A})$ can be constructed by providing $\mathcal{A}$ with a $Rep(H \times \mathbb{G}_m)$ action by exact functors, introducing an R linear structure on $\mathcal{A}_{deeq}$ where $R = \mathcal{O}(C)$ is the homogeneous coordinate ring of

the projective variety $\overline{X}$, and verifying that the resulting S' linear structure sends $D_{perf}(S')_{\partial S'}$ to zero.

Remark 3. Most of the statements in the main Theorem of the paper assert an equivalence between (a subcategory of) $D^b(Coh(S))$ for an algebraic stack S and $D^b(\mathcal{A})$ for an abelian category $\mathcal{A}$ (with the exception of (4) which involves coherent sheaves on a DG-stack and an equivariant derived category of constructible sheaves).

We first construct the S-linear structure on $\mathcal{C} = D^b(\mathcal{A})$ and then consider the action on a particular object of $\mathcal{C}$ to get an equivalence. The construction of the action almost follows the pattern of example (5). The difference is as follows. We have $S = X/G\check{}$ where X admits an affine equivariant map to $\mathcal{B}^2$. Though $\mathcal{B}^2$ is a projective variety there is no preferred choice of an equivariant projective embedding, so to keep things more canonical we work with the "multi-homogeneous" coordinate ring and consider open embeddings of our stacks into $Y/(G\check{}\times T\check{}^2)$ for an appropriate affine variety Y. A more essential difference is that while $Rep(G\check{})$ acts by exact functors on our abelian category $\mathcal{A}$, the action of $Rep(T\check{}^2)$ is only defined on the triangulated category $\mathcal{C}$, it is not compatible with the natural t-structure on $\mathcal{C} = D^b(\mathcal{A})$.

An additional argument based on properties of tilting modules is needed to deal with this issue (see subsection 4.4.2).

2.2.2. *The list of structures.* We concentrate on the equivalence (2), the equivalence (3) is similar, and (4) will be deduced formally from (2)

Consider the following sequence of maps

$$St/G\check{} \rightrightarrows \tilde{\mathfrak{g}}\check{}/G\check{} \to \mathfrak{g}\check{}/G\check{} \to pt/G\check{}.$$

Moving from right to left in this sequence, we successively equip $\hat{D}$ with the linear structure for the corresponding stack.

The $pt/G\check{}$ linear structure comes from an action of the tensor category $Rep(G\check{})$ on the abelian category $\mathcal{P}_{I^0,I}$. Such an action was defined in [17] where the *central sheaves* categorifying the canonical basis in the center of the affine Hecke algebra were constructed; an extension of the action to $\mathcal{P}_{I^0,I^0}$ is sketched in section 3.5 below.

By a version of Tannakian formalism, lifting the action of the tensor category $Rep(G\check{})$ action to a $\mathfrak{g}\check{}/G\check{}$ structure amounts to equipping the with a tensor endomorphism. Such an endomorphism comes from the *logarithm of monodromy* acting on central sheaves: recall that the central sheaves are constructed by nearby cycles which carry a monodromy automorphism.

We now discuss the two structures of a stack over $\tilde{\mathfrak{g}}\check{}/G\check{}$. The starting point here is the familiar observation that for a representation V of $G\check{}$ the trivial vector bundle $V \otimes \mathcal{O}_{\mathcal{B}}$ with fiber V on the flag variety $\mathcal{B} = G\check{}/B\check{}$ carries a canonical filtration whose associated graded is a sum of line bundles. This filtration can be lifted to a filtration can lifted to a similar filtration for $V \otimes \mathcal{O}_{\tilde{\mathfrak{g}}\check{}}$. Under our equivalences this filtration corresponds to a filtration on (monodromic) central sheaves by (monodromic) *Wakimoto sheaves* (the non-monodromic version was presented in [1], and the monodromic generalization is presented below in section 3). It turns out that the filtration defines a monoidal functor $D^b(Coh^{G\check{}}(\tilde{\mathfrak{g}}\check{})) \to \hat{D}$, where $\hat{D}$ is a certain completion of D_{I^0,I^0} (see below); we then get two commuting actions of $D^b(Coh^{G\check{}}(\tilde{\mathfrak{g}}\check{}))$ on D_{I^0,I^0} from the left and the right action of the monoidal category

$\hat{D}$ on itself. Since $Rep(G\check{})$ acts by central functors and the tensor endomorphism is compatible with the central structure, we get the $St/G\check{}$-linear structure where $St = \tilde{\mathfrak{g}}\check{} \times_{\mathfrak{g}\check{}} \tilde{\mathfrak{g}}\check{}$ is the fiber square.

More precisely, we get the monoidal functor $D^b(Coh^{G\check{}}(\tilde{\mathfrak{g}}\check{})) \to \hat{D}$ from the filtration following a strategy similar to the one in Example (5) above. The first term of the filtration (the "lowest weight arrow") determines a functor from $D^b_{perf}(Coh^{G\check{}\times T\check{}}(C))$ where C is a certain affine scheme with an action of $G\check{} \times T\check{}$ with an open $G\check{} \times T\check{}$ equivariant embedding $G\check{}/U\check{} \to C$. The fact that the lowest weight arrow extends to a filtration satisfying certain properties implies that complexes supported on $\partial G\check{}/U\check{} = C \setminus G\check{}/U\check{}$ act by zero. These ideas have already been used in [1].

Once the $St/G\check{}$ linear structure on $\hat{D}$ is constructed, any object $M \in \hat{D}$ defines a functor $D^{G\check{}}_{perf}(St) \to \hat{D}$, $\mathcal{F} \mapsto \mathcal{F}(M)$. We use the functor (denoted by Φ_{perf}) corresponding to the choice $M = \hat{\Xi}$ where $\hat{\Xi}$ is a certain tilting pro-object discussed in section 5. This choice can be motivated by the requirement of compatibility with the equivalence $\Phi^{I_0}_{IW}$: the object $\hat{\Xi}$ is obtained from the unit object in $\hat{D}$ by projection to $\hat{D}_{IW}$ composed with its adjoint, on the dual side this corresponds to the sheaf $pr^*_{Spr,2} pr_{Spr,2*}(\delta_*(\mathcal{O}_{\tilde{\mathfrak{g}}\check{}})) \cong \mathcal{O}_{St}$, where $\delta : \tilde{\mathfrak{g}}\check{} \to St$ is the diagonal embedding (where we omitted completion at the preimage of $\mathcal{N}$ from notation). Thus the compatibility implies that $\Phi_{perf}(\mathcal{O}) \cong \hat{\Xi}$.

The fact that Φ_{perf} constructed this way is compatible with projection to D^I_{IW} follows from the properties of Ξ.

We then establish the equivalence $\Phi^{I_0}_{IW}$ as in [1]. Together with compatibilities between Φ_{perf} and $\Phi^{I_0}_{IW}$ this implies that Φ_{perf} is a full embedding.

Once Φ_{perf} is constructed we deduce an equivalence Ψ from a general result relating the categories $D^b(Coh(X))$ and $D_{perf}(X)$ for an algebraic stack X. We show that $D^b(Coh(X))$ embeds into the category of functors $D_{perf}(X) \to Vect$ and characterize the image of this embedding. The characterization makes use of the standard t-structure on the derived category of coherent sheaves. In order to apply the general criterion in our situation we show that, although the functor Φ_{perf} is not t-exact with respect to the natural t-structures on the two triangulated categories, it satisfies a weaker compatibility (see section 8).

Thus we obtain an equivalence Ψ and the inverse equivalence Φ.

Monoidality is then deduced from compatibility with the action on $D^b(Coh^{G\check{}}_{\mathcal{N}}(\tilde{\mathfrak{g}}\check{})) \cong D^{I^0}_{IW}$. We use presentation of $\hat{D}$ as homotopy category of complexes of free-monodromic tilting sheaves introduced in [13] and recalled below. Using the observation that convolution of two free monodromic tilting sheaves is also a free monodromic tilting sheaf we get an explicit monoidal structure on the category of tilting complexes, which is identifed with the monoidal structure on $\hat{D}$. The monoidal structure on the equivalences Φ, Ψ follows from compatibility with the action on $\hat{D}_{IW}$, since a sheaf in $D^b(Coh^{G\check{}}(\widehat{St}))$ can be uniquely reconstructed from the endo-functor of $D^b(Coh^{G\check{}}_{\mathcal{N}}(\tilde{\mathfrak{g}}\check{}))$ given by convolution with $\mathcal{F}$.

2.3. **Description of the content.** Sections 3 and 5 mostly recall the results of [13] while section 4 recalls the material of [1] and extends it to the present slightly more general setting.

It is technically convenient to enlarge both categories in (2) and construct the equivalence

$$\widehat{D}_{I^0,I^0} \cong D^b(Coh^{G^\check{}}(\widehat{\tilde{\mathfrak{g}}^\check{} \times_{\mathfrak{g}^\check{}} \tilde{\mathfrak{g}}^\check{}})) \tag{8}$$

where $\widehat{\tilde{\mathfrak{g}}^\check{} \times_{\mathfrak{g}^\check{}} \tilde{\mathfrak{g}}^\check{}}$ denotes the formal completion of $\tilde{\mathfrak{g}}^\check{} \times_{\mathfrak{g}^\check{}} \tilde{\mathfrak{g}}^\check{}$ at the preimage of $\mathcal{N}$ and $\widehat{D}_{I^0,I^0}$ (abbreviated as $\hat{D}$) is a certain subcategory in the category of pro-objects in D_{I^0,I^0}. In section 3 we recall the definition of $\widehat{D}$ and an extension of the formalism of tilting sheaves to this settting. We also present a "monodromic" generalization of central sheaves [17].

Section 4 provides a generalization of the main result of [1] to the monodromic setting. Namely, it establishes a monoidal functor F from the derived category of equivariant coherent sheaves on the formal completion of $\tilde{\mathfrak{g}}^\check{}$ at $\tilde{\mathcal{N}}$ to $\widehat{D}_{I^0,I^0}$. (The composition of this functor with the equivalence (8) which will be established later is the direct image under the diagonal embedding $\tilde{\mathfrak{g}}^\check{} \to \tilde{\mathfrak{g}}^\check{} \times_{\mathfrak{g}^\check{}} \tilde{\mathfrak{g}}^\check{}$.) A variation of the argument allows us to define the action of the tensor category $(D_{perf}(Coh^{G^\check{}}(St)), \otimes_{\mathcal{O}})$ on D_{I^0,I^0} and $\hat{D}$.

We also consider the projection of $\widehat{D}_{I^0,I^0}$ to the Iwahori-Whittaker category $\hat{D}_{IW}$ and show that the composition of F with this projection induces an equivalence $D^b(Coh^{G^\check{}}_{\mathcal{N}}(\tilde{\mathfrak{g}}^\check{})) \cong \hat{D}_{IW}$; here $\hat{D}_{IW}$ is a certain category of pro-objects in $D^{I^0}_{IW}$.

Section 5 is devoted to a particular object $\hat{\Xi} \in \widehat{\mathcal{P}}_{I^0,I^0}$ which will correspond to the structure sheaf of $\widehat{\tilde{\mathfrak{g}}^\check{} \times_{\mathfrak{g}^\check{}} \tilde{\mathfrak{g}}^\check{}}$ under the equivalence.

Then in section 6 we define a functor Φ_{perf} from the subcategory of perfect complexes $D^b_{perf}(Coh^{G^\check{}}(\widehat{\tilde{\mathfrak{g}}^\check{} \times_{\mathfrak{g}^\check{}} \tilde{\mathfrak{g}}^\check{}})) \subset D^b(Coh^{G^\check{}}(\widehat{\tilde{\mathfrak{g}}^\check{} \times_{\mathfrak{g}^\check{}} \tilde{\mathfrak{g}}^\check{}}))$ to $D^b(\widehat{\mathcal{P}}_{I^0,I^0})$ by applying the action of $D_{perf}(Coh^{G^\check{}}(St)), \otimes_{\mathcal{O}}$ on Ξ.

We use properties of Ξ to show that Φ_{perf} is compatible with the action of the monoidal category $D^b_{perf}(Coh^{G^\check{}}(\widehat{\tilde{\mathfrak{g}}^\check{} \times_{\mathfrak{g}^\check{}} \tilde{\mathfrak{g}}^\check{}}))$ on the module category $D^b(Coh^{G^\check{}}(\hat{\tilde{\mathfrak{g}}}^\check{}))$ under the equivalence Φ^f. This allows us to deduce that Φ_{perf} is a full embedding and endow it with the structure of a monoidal functor.

In section 8 we check a property of Φ with respect to the natural t-structures on the two categories. In the next section 7 we give a general criterion allowing to extend an equivalence from the category of perfect complexes to the bounded category of coherent sheaves.

In section 9 we show that criterion of section 7 applies, by virtue of properties established in section 8, to the present situation yielding (8). We then deduce (3) and (4) by means of a general lemma describing the equivariant constructible category via the monodromic one.

3. Monodromic sheaves and pro-object

3.1. **Generalities on monodromic sheaves.** Objects of $\mathcal{P}_{I^0,I^0}$ are by definition perverse sheaves monodromic with respect to both the left and the right action of T on $\widetilde{\mathcal{F}\ell}$. Thus the group $\Lambda \times \Lambda$ acts on $\mathcal{P}_{I^0,I^0}$ by automorphisms of identity functor, the action on each object is unipotent.

Let $\hat{\mathcal{P}}$ be the category of pro-objects M in $\mathcal{P}_{I^0,I^0}$ such that the coinvariants of monodromy M_{mon} belongs to $\mathcal{P}$. Set $\hat{D} = D^b(\hat{\mathcal{P}})$; according to [13], $\hat{D}$ can be identified with a full subcategory in the category of pro-objects in D_{I^0,I^0}. Furthermore, $\hat{D}$ is monoidal and contains D_{I^0,I^0} as a full tensor subcategory (see [**?**, BY].

An object $\mathcal{F} \in \hat{D}$ belongs to D_{I^0,I^0} iff the monodromy automorphisms of $\mathcal{F}$ are unipotent.

Let $\mathcal{E}$ be the *free prounipotent* rank one local system on T (see [13]), thus $\mathcal{E} = \varprojlim \mathcal{E}_n$ where $\mathcal{E}_n$ is the local system whose fiber at the unit element $1_T \in T$ is identified with the quotient of the group algebra of tame fundamental group $\pi_1^{tame}(T)$ by the n-th power of augmentation ideal, where the action of monodromy coincides with the natural structure of $\pi_1^{tame}(T)$ module. The quotient $\mathbf{I}^0\backslash\widetilde{\mathcal{F}\ell}_w$ is a torsor over T, choosing an arbitrary trivialization of the torsor we get a project $\widetilde{\mathcal{F}\ell}_w \to T$ which we denote pr_w. Set $\Delta_w = j_{w!}pr_w^*(\mathcal{E})[\dim \widetilde{\mathcal{F}\ell}_w]$, $\nabla_w = j_{w*}pr_w^*(\mathcal{E})[\dim \widetilde{\mathcal{F}\ell}_w]$. The objects Δ_w, ∇_w are defined uniquely up to a non-unique isomorphism, we call them a free-monodromic standard and costandard objects respectively.

3.2. **More on monodromic (co)standard pro-sheaves.** The free prounipotent local system $\mathcal{E}$ is defined uniquely up to a non-unique isomorphism, thus so are the (co)standard sheaves Δ_w, ∇_w.

For $w = \lambda \in \Lambda \subset W$ the choice of a uniformizer $t \in F$ and a maximal torus $\mathbf{T} \subset G$ which is in good relative position with $\mathbf{I}$ define an element $t_\lambda = \lambda(t) \in T_F \subset G$; its image in $\widetilde{\mathcal{F}\ell} = \mathbf{G}/\mathbf{I}^0$ lies in the orbit of I corresponding to λ. This yields the choice of a point $\overline{\lambda(t)} \in \mathbf{I}^0\backslash\widetilde{\mathcal{F}\ell}_\lambda$ which gives a trivialization of the T-torsor, and hence the choice of objects Δ_λ, ∇_λ defined uniquely up to a unique isomorphism. [We use the same notation for those canonically defined objects and the objects defined earlier uniquely up to a non-unique isomorphism].

Lemma 4. *a) We have isomorphisms $\Delta_{w_1} * \Delta_{w_2} \cong \Delta_{w_1w_2}$, $\nabla_{w_1} * \nabla_{w_2} \cong \nabla_{w_1w_2}$ when $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$.*

b) Assume that $w_1 = \lambda_1$, $w_2 = \lambda_2 \in \Lambda^+$ and let Δ_{λ_i}, ∇_{λ_i}, $(i = 1, 2)$ be the canonically defined objects as above. We have canonical isomorphisms s $\Delta_{\lambda_1} * \Delta_{\lambda_2} \cong \Delta_{\lambda_1+\lambda_2}$, $\nabla_{\lambda_1} * \nabla_{\lambda_2} \cong \nabla_{\lambda_1+\lambda_2}$, which satisfies the associativity identity for a triple λ_1, λ_2, λ_3.

c) $\Delta_0 = \nabla_0$ is the unit object in $\hat{D}$; and we have a canonical isomorphism $\nabla_w * \Delta_{w^{-1}} \cong \nabla_0$.

d) We have $\Delta_{w_1} * \nabla_{w_2} \in \hat{\mathcal{P}}$, $\nabla_{w_1} * \Delta_{w_2} \in \hat{\mathcal{P}}$ for all $w_1, w_2 \in W$.

e) $\pi_*(\nabla_w) \cong j_{w*}$, $\pi_*(\Delta_w) \cong j_{w!}$ canonically.

Proof. (a,d) and a noncanonical isomorphism in (c) are shown in [26] Given λ_1, $\lambda_2 \in \Lambda^+$ consider the locally closed subvariety in the convolution diagram: $\widetilde{\mathcal{F}\ell}_{\lambda_1} \boxtimes^{\mathbf{I}^0} \widetilde{\mathcal{F}\ell}_{\lambda_2} \to \widetilde{\mathcal{F}\ell}_{\lambda_1+\lambda_2}$. Using the above trivializations of the T torsors $\mathbf{I}^0\backslash\widetilde{\mathcal{F}\ell}_{\lambda_1}$, $\mathbf{I}^0\backslash\widetilde{\mathcal{F}\ell}_{\lambda_2}$, $\mathbf{I}\backslash\widetilde{\mathcal{F}\ell}_{\lambda_1+\lambda_2}$ we can identify the quotient of $\widetilde{\mathcal{F}\ell}_{\lambda_1} \boxtimes^{\mathbf{I}^0} \widetilde{\mathcal{F}\ell}_{\lambda_2}$ by $\mathbf{I}^0$ with $T \times T$ and the quotient of $\widetilde{\mathcal{F}\ell}_{\lambda_1+\lambda_2}$ by $\mathbf{I}^0$ with T; the quotient of the convolution map is readily seen to be the multiplication map $T \times T \to T$. Since the convolution $\mathcal{E} *_T \mathcal{E}$ is canonically isomorphic to $\mathcal{E}$ (here $*_T$ denotes convolution of sheaves on the group T) we get the desired canonical isomorphism. Verification of the associativity identity is straightforward.

Finally, part (e) easily follows from the fact that cohomology of the free prounipotent local system on T is zero in degrees other than $r = \dim(T)$ and r-th cohomology is one dimensional. $\square$

3.3. **Wakimoto pro-sheaves.** Recall *Wakimoto sheaves* $J_\lambda \in \mathcal{P}_{I,I}$ characterized by: $J_\lambda * J_\mu \cong J_{\lambda+\mu}$ for $\lambda, \mu \in \Lambda$ and $J_\lambda = j_{\lambda *}$ for $\lambda \in \Lambda_+$, see [1, 3.2]. The following monodromic version follows directly from Lemma 4(b,c).

Corollary 5. *There exists a monoidal functor $\Theta : Rep(T\check{}) \to \hat{D}$ sending a dominant character λ to ∇_λ and an anti-dominant character μ to ∇_μ. Such a functor is defined uniquely up to a unique isomorphism.* □

The image of a character λ of $T\check{}$ under this functor will be called a free monodromic Wakimoto sheaf and will be denoted by $\mathcal{J}_\lambda$.

Some of the basic properties of Wakimoto sheaves are as follows.

Lemma 6. *We have:*
a) $\mathcal{J}_\lambda \in \hat{\mathcal{P}} \subset \hat{D}$.
b) $Hom^\bullet(\mathcal{J}_\lambda, \mathcal{J}_\mu) = 0$ for $\mu \not\preceq \lambda$.
c) $\pi_*(\mathcal{J}_l a) \cong J_\lambda$ canonically.

Proof. a) follows from Lemma 4(d). b) is clear since

$$Hom^\bullet(\mathcal{J}_\lambda, \mathcal{J}_\mu) \cong Hom^\bullet(\mathcal{J}_{\lambda+\eta}, \mathcal{J}_{\mu+\eta}) = Hom(\nabla_{\lambda+\eta}, \nabla_{\mu+\eta}),$$

where $\eta \in \Lambda$ is chosen so that $\lambda+\eta$, $\mu+\eta \in \Lambda^+$. The latter Hom space vanishes when $\lambda \not\preceq \mu$ because in this case $\widetilde{\mathcal{F}\ell}_{\mu+\eta}$ is not contained in the closure of $\widetilde{\mathcal{F}\ell}_{\lambda+\eta}$. Part (c) follows from Lemma 4(e). □

3.4. **Generalized tilting pro-objects.** Recall that an object of $\mathcal{P}$ is called *tilting* if it carries a standard and also a costandard filtration; here a filtration is called (co)standard if its associated graded is a sum of (co)standard objects, see e.g. [5].

An object of $\hat{\mathcal{P}}$ is called *free-monodromic tilting* if it carries a free-monodromic standard and also a free-monodromic costandard filtrations; here a filtration is called (co)standard if its associated graded is a sum of free-monodromic (co)standard objects, see [13].

Let $\mathcal{T} \subset \mathcal{P}$ be the full subcategory of tilting objects and $\hat{\mathcal{T}} \subset \hat{\mathcal{P}}$ denote the full subcategory of free-monodromic tilting objects [13].

Let $Com(\mathcal{T})$, $Com(\hat{T})$ denote the homotopy category of bounded complexes of objects in $\mathcal{T}$, $\hat{T}$ respectively.

The next Proposition summarizes the properties of tilting objects that will be used in the argument.

Proposition 7. *a) The natural functors $Com(\mathcal{T}) \to D^b(\mathcal{P}) = D$, $Com(\hat{T}) \to D^b(\hat{\mathcal{P}}) = \hat{D}$ are equivalences.*

b) The convolution of two object in $\hat{T}$ lies in $\hat{T}$, thus $Com(\hat{T})$ has a natural monoidal structure. The natural functor $Com(\hat{T}) \to \hat{D}$ is a monoidal equivalence.

Proof. The first statement in (a) appears in [5], the second one (whose proof is similar) is a particular case of [13, Proposition B.1.7].

The two statements in part (b) follow from [13, Proposition 4.3.4] and [13, Proposition B.3.1, Remark B.3.2] respectively. □

Remark 8. Implicit in Proposition 7(a) is Ext vanishing:

$$Ext^{>0}(\hat{T}_1, \hat{T}_2) = 0 = Ext^{>0}(T_1, T_2)$$

for $T_1, T_2 \in \hat{\mathcal{T}}$, $\hat{T}_1, \hat{T}_2 \in \hat{\mathcal{T}}$. A stronger statement will be used later:

$$Ext^{>0}(\widehat{M}_1, \widehat{M}_2) = 0 = Ext^{>0}(M_1, M_2)$$

where M_1, $M_2 \in \mathcal{P}_{\mathcal{I}^0,I}$, M_1 admits a standard filtration while M_2 admits a costandard filtration, $\widehat{M}_1$, $\widehat{M}_2 \in \hat{\mathcal{P}}$, M_1 admits a free-monodromic standard filtration, while M_2 admits a free-monodromic costandard filtration. The proof is immediate from $Ext^{>0}(\Delta_{w_1}, \nabla_{w_2}) = 0 = Ext^{>0}(j_{w_1!}, j_{w_2*})$.

Proposition 9. *An object $M \in \hat{D}$ admits a free-monodromic (co)standard filtration iff $\pi_*(M) \in D_{I^0,I}$ lies in $\mathcal{P}$ and admits a (co)standard filtration.*

Proof. The "only if" direction follows from Lemma 4(e), while the "if" direction is checked in [13]. □

Corollary 10. *An object $M \in \hat{D}$ lies in $\hat{\mathcal{T}}$ iff $\pi_*(M) \in \mathcal{T}$.*

Proposition 11. [13] *a) For $T \in \hat{\mathcal{T}}$ the functors $\mathcal{F} \mapsto T * \mathcal{F}$ and $\mathcal{F} \mapsto \mathcal{F} * T$ are t-exact (i.e. send $\mathcal{P}_{I^0,I^0}$ to $\mathcal{P}_{I^0,I^0}$ and $\hat{\mathcal{P}}$ to $\hat{\mathcal{P}}$).*

b) For any $w \in W$ there exists a unique (up to an isomorphism) indecomposable object $T_w \in \mathcal{T}$ whose support is the closure of $\mathcal{F}\ell_w$. There also exists a unique indecomposable object $\hat{T}_w \in \hat{\mathcal{T}}$ whose support is the closure of $\widetilde{\mathcal{F}\ell}_w$. We have $\pi_*(\hat{T}_w) \cong T_w$.

c) For $T \in \hat{\mathcal{T}}$ and $w \in W$ the objects $\Delta_w * T$, $T * \Delta_w \in \hat{\mathcal{P}}$ have a free-monodromic standard filtration, while the objects $\nabla_w * T$, $T * \nabla_w \in \hat{\mathcal{P}}$ have a free-monodromic costandard filtration. □

Corollary 12. *Convolution with a monodromic free tilting object preserves the categories of objects admitting a free-monodromic (co)standard filtration (same with monodromic free).*

3.5. **Monodromic central sheaves.** We need to extend the central functors of [17] to the monodromic setting.

3.5.1. *A brief summary of* [17]. Recall first the main result of [17]. In our present notation it reads as follows.

For $V \in Rep(\check{G})$ one defines an exact functor $\mathcal{Z}_V : \mathcal{P}_{I,I} \to \mathcal{P}_{I,I}$. The functors come with canonical isomorphisms

One then constructs canonical isomorphisms

$$Z_V * \mathcal{F} \cong Z_V(\mathcal{F}) \cong \mathcal{F} * Z_V, \quad \mathcal{F} \in \mathcal{P}_{I,I}; \tag{9}$$

$$Z_{V \otimes W} \cong Z_V \circ Z_W, \tag{10}$$

where $Z_V = \mathcal{Z}_V(\delta_e)$.

The two isomorphisms satisfy natural compatibilities (some are demonstrated in [**?**]) which amount to saying that $V \mapsto Z_V$ is a tensor functor from $Rep(\check{G})$ to Drinfeld center of $D_{I,I}$.

The goal of this subsection is to extend these results to the monodromic setting.

Construction of the functor Z_V is based on existence of a certain deformation of the affine flag variety $\mathcal{F}\ell$ and the convolution diagrams.

Let C be an algebraic curve over k and fix a point $x_0 \in C(k)$ and set $C_0 = C \setminus \{x_0\}$. The ind-schemes $\mathcal{F}\ell_C^{(2)}$, $Conv_C$, $Conv'_C$ were constructed in [17]. The

ind-schemes $Fl^{(2)}_C$, $Conv_C$, $Conv'_C$ come with a map to C satisfying the following properties.

The preimage of x_0 in $\mathcal{F}\ell^{(2)}_C$ is identified with $\mathcal{F}\ell$, while the preimage of $C\backslash\{x_0\}$ is identified with $\mathcal{F}\ell\times\mathcal{G}\mathfrak{r}_{C_0}$ where $\mathcal{G}\mathfrak{r}_{C_0}$ is the *Beilinson-Drinfeld global Grassmannian*; thus the fiber of $\mathcal{F}\ell^{(2)}_C$ over $y\in C_0(k)$ is (noncanonically) isomorphic to $\mathcal{F}\ell\times\mathcal{G}\mathfrak{r}$.

To spell out the properties of $Conv_C$, $Conv'_C$ recall the convolution space $\mathcal{F}\ell\times^{\mathbf{I}}\mathcal{F}\ell$, which is the fibration over $\mathcal{F}\ell$ with fiber $\mathcal{F}\ell$ associated with the natural principal $\mathbf{I}$ bundle over $\mathcal{F}\ell$ using the action of $\mathbf{I}$ on $\mathcal{F}\ell$. We have the projection map $pr_1 : \mathcal{F}\ell\times^{\mathbf{I}}\mathcal{F}\ell\to\mathcal{F}\ell$ and the convolution map $conv:\mathcal{F}\ell\times^{\mathbf{I}}\mathcal{F}\ell\to\mathcal{F}\ell$ coming from multiplication map of the group $\mathbf{G_F}$.

The fiber of both $Conv_C$ and $Conv'_C$ over x_0 is $\mathcal{F}\ell\times^{\mathbf{I}}\mathcal{F}\ell$; the preimage of C^0 in $Conv_C$ is the product of $((G/B)\times^{\mathbf{I}}\mathcal{F}\ell)\times\mathcal{G}\mathfrak{r}_{C^0}$, while the preimage of C^0 in $Conv'_C$ is identified with $(\mathcal{F}\ell\times^{\mathbf{I}}(G/B))\times\mathcal{G}\mathfrak{r}_{C^0}$.

One has canonical ind-proper morphisms $conv_C : Conv_C\to\mathcal{F}\ell^{(2)}$, $conv'_C : Conv'_C\to\mathcal{F}\ell^{(2)}$ whose fiber over x_0 is the convolution map $conv$.

Starting from $V\in Rep(G\check{\ })$ one can use the geometric Satake of isomorphism to produce a semi-simple perverse sheaf $S(V)$ on Gr_{C^0}. For $\mathcal{F}\in Perv(\mathcal{F}\ell)$ one gets a sheaf $\mathcal{F}\boxtimes S(V)$ on $\mathcal{F}\ell\times Gr_{C^0}\subset\mathcal{F}\ell^{(2)}_C$. Taking nearby cycles of that sheaf with respect to a local coordiante at x_0 one obtains a sheaf $\mathcal{Z}_V(\mathcal{F})$ on $\mathcal{F}\ell$.

The spaces $Conv_C$, $Conv'_C$ and the maps $conv_C$, $conv'_C$ are used in [17] to show that the functor $\mathcal{Z}_V|_{D_{I,I}}$ is isomorphic to both left and right convolution with a certain object $Z_V\in\mathcal{P}_I$.

3.5.2. *The monodromic case.* A straightforward modification of the definition from [17] yields spaces $\widetilde{\mathcal{F}\ell}^{(2)}_C$, $\widetilde{Conv}_C$, $\widetilde{Conv}'_C$ with the following properties.

The ind-schemes $\widetilde{\mathcal{F}\ell}^{(2)}_C$, $\widetilde{Conv}_C$, $\widetilde{Conv}'_C$ come with a map to C satisfying the following properties.

The preimage of x_0 in $\widetilde{\mathcal{F}\ell}^{(2)}_C$ is identified with $\widetilde{\mathcal{F}\ell}$, while the preimage of $C\backslash\{x_0\}$ is identified with $\widetilde{\mathcal{F}\ell}\times\mathcal{G}\mathfrak{r}_{C_0}$ where $\mathcal{G}\mathfrak{r}_{C_0}$ is the *Beilinson-Drinfeld global Grassmannian*; thus the fiber of $\widetilde{\mathcal{F}\ell}^{(2)}_C$ over $y\in C_0(k)$ is (noncanonically) isomorphic to $\widetilde{\mathcal{F}\ell}\times\mathcal{G}\mathfrak{r}$.

We will now use the convolution space $\widetilde{\mathcal{F}\ell}\times^{\mathbf{I}^0}\widetilde{\mathcal{F}\ell}$, which is a fibration over $\widetilde{\mathcal{F}\ell}$ with fiber $\widetilde{\mathcal{F}\ell}$ associated with the natural principal $\mathbf{I}$ bundle over $\widetilde{\mathcal{F}\ell}$ using the action of $\mathbf{I}^0$ on $\widetilde{\mathcal{F}\ell}$. We have the projection map $pr_1:\widetilde{\mathcal{F}\ell}\times^{\mathbf{I}^0}\widetilde{\mathcal{F}\ell}\to\widetilde{\mathcal{F}\ell}$ and the convolution map $conv_{I^0}:\widetilde{\mathcal{F}\ell}\times^{\mathbf{I}^0}\widetilde{\mathcal{F}\ell}\to\widetilde{\mathcal{F}\ell}$ coming from multiplication map of the group $\mathbf{G_F}$.

The fiber of both $\widetilde{Conv}_C$ and $\widetilde{Conv}'_C$ over x_0 is $\widetilde{\mathcal{F}\ell}\times^{\mathbf{I}^0}\widetilde{\mathcal{F}\ell}$; the preimage of C^0 in $\widetilde{Conv}_C$ is the product of $((G/U)\times^{\mathbf{I}^0}\widetilde{\mathcal{F}\ell})\times\mathcal{G}\mathfrak{r}_{C^0}$, while the preimage of C^0 in $\widetilde{Conv}'_C$ is identified with $(\widetilde{\mathcal{F}\ell}\times^{\mathbf{I}}(G/U))\times\mathcal{G}\mathfrak{r}_{C^0}$.

One has canonical morphisms $\widetilde{conv}_C:\widetilde{Conv}_C\to\widetilde{\mathcal{F}\ell}^{(2)}$, $\widetilde{conv}'_C:\widetilde{Conv}'_C\to\widetilde{\mathcal{F}\ell}^{(2)}$ whose fiber over x_0 is the convolution map $\widetilde{conv}$.

The main technical difference with the setting of [17] recalled in the previous subsection is that in contrast with the maps $conv_C$, $conv'_C$ the maps $\widetilde{conv}_C$, $\widetilde{conv}'_C$ are *not ind-proper*.

For $V \in Rep(G\check{})$ and $\mathcal{F} \in D$ we can form a complex $\mathcal{F} \boxtimes S(V)$ on $\widetilde{\mathcal{F}\ell} \times \mathcal{G}\mathfrak{r}_{C^0} \subset \widetilde{\mathcal{F}\ell}_C^{(2)}$. Taking nearby cycles with respect to a local coordinate on C near x_0 we get a complex which we denote $\hat{\mathcal{Z}}_V(\mathcal{F})$.

The functor $\hat{\mathcal{Z}}_V$ obviously extends to the category $\hat{D}$. We set $\hat{Z}_V = \hat{\mathcal{Z}}_V(\Delta_e)$.

Proposition 13. *a) Recall that $\pi : \widetilde{\mathcal{F}\ell} \to \mathcal{F}\ell$ is the projection. Then we have $\hat{\mathcal{Z}}_V(\pi^*\mathcal{F}) \cong \pi^*(\mathcal{Z}_V(\mathcal{F}))$ canonically.*

b) $\hat{\mathcal{Z}}_V$ is canonically isomorphic to the functors of both left and right convolution with $\hat{Z}_V$.

c) The map $V \mapsto \hat{Z}_V$ extends to a central functor $Rep(G\check{}) \to \hat{D}$, i.e. to a tensor functor from $Rep(G\check{})$ to the Drinfeld center of $\hat{D}$.

d) We have a canonical isomorphism $\pi_*(\hat{Z}_V) \cong Z_V$.

Proof. a) follows from the fact that nearby cycles commute with pull-back under a smooth morphism.

The proof of (b,c) is parallel to the argument of [17] and [**?**] respectively, with the following modification. The argument of *loc. cit.* uses that the convolution maps and its global counterparts (denoted presently by $conv_C$, $conv'_C$) are proper in order to apply the fact that nearby cycles commute with direct image under a proper map. The maps $\widetilde{conv}$, $\widetilde{conv}_C$, $\widetilde{conv}'_C$ are not proper, thus we do not a'priori have an isomorphism between the direct image under $\widetilde{conv}_C$ or $\widetilde{conv}'_C$ of nearby cycles of a sheaf and nearby cycles of its direct image. However, we do have a canonical map in one direction. If we start from a sheaf on $\widetilde{\mathcal{F}\ell}$ which is the pull-back of a sheaf on $\mathcal{F}\ell$, then the map is an isomorphism because the sheaves in question are pull-backs under a smooth map of ones considered in [17]. Since all objects of D can be obtained from objects in the image of the pull-back functor $D_{I^0,I^0} \to D$ by successive extensions, the map in question is an isomorphism for any $\mathcal{F} \in D$, and claims (b,c) follows.

d) follows from (a). □

3.5.3. *Monodromy endomorphisms.* Being defined as (the inverse limit of) nearby cycles sheaves, the objects $\hat{Z}_V$, $V \in Rep(G\check{})$ carry a canonical *monodromy automorphism.* It is known that the monodromy automorphism acting on the sheaf Z_V is unipotent, it follows that the one acting on $\hat{Z}_V$ is pro-unipotent. We let $m_V : \hat{Z}_V \to \hat{Z}_V$ denote the logarithm of monodromy.

It will be useful to have an alternative description of this endomorphism. Consider the action of $\mathbb{G}_m$ on $\widetilde{\mathcal{F}\ell}$ by *loop rotation.* Since each $\mathbf{I} \times \mathbf{I}$ orbit on $\widetilde{\mathcal{F}\ell}$ is invariant under this action, every object of $\mathcal{P}_{I^0,I^0}$ is $\mathbb{G}_m$ monodromic with unipotent monodromy. Thus every $\mathcal{F} \in \mathcal{P}_{I^0,I^0}$ acquires a canonical logarithm of monodromy endomorphism which we denote by $\mu_{\mathcal{F}}$. By passing to the limit we also get a definition of $\mu_{\mathcal{F}}$ for $\mathcal{F} \in \hat{\mathcal{P}}$.

Proposition 14. *a) We have $m_V = \mu_{\hat{Z}_V}$.*

b) The logarithm of monodromy defines a tensor endomorphism of the functor $\hat{Z}$, i.e. we have $m_{V\otimes W} = m_V * Id_{\hat{Z}_W} + Id_{Z_V} * m_W$.

Proof. a) follows by the argument of [1, 5.2], while (b) is parallel to [17, Theorem 2]. □

3.5.4. *Filtration of central sheaves by Wakimoto sheaves.* It will be convenient to fix a total ordering on Λ compatible with addition and the standard partial order. This allows to make sense of an object in an abelian category with a filtration indexed by Λ and of its associated graded.

Recall that the object $\mathcal{J}_\lambda$ was defined canonically up to a unique isomorphism starting from a fixed uniformizer t of the local field F, while the central functor $\mathcal{Z}_V$ was defined using an algebraic curve C with a point x_0 together with a fixed isomorphism between F and the field of functions on the punctured formal neighborhood of x_0 in C together a fixed etale local coordinate at x_0. In the next Proposition we assume that t is given by the local coordinate.

Proposition 15. *a) For any λ there exists a canonical injective morphism ϖ_λ : $\hat{Z}_\lambda \to \mathcal{J}_\lambda$. It is compatible with convolution in the following way: the composition of $\varpi_{\lambda+\mu}$ with the canonical map $\hat{Z}_\lambda * \hat{Z}_\mu \to \hat{Z}_{\lambda+\mu}$ equals $\varpi_\lambda * \varpi_\mu$.*

b) The surjection ϖ_λ extends to a unique filtration on $\hat{Z}_\lambda$ indexed by Λ with associated graded isomorphic to a sum of Wakimoto sheaves $\mathcal{J}_\mu$.

c) The filtration on $\hat{Z}_\lambda$ is compatible with the monoidal structure on the functor $V \mapsto \hat{Z}_V$, making $V \mapsto gr(\hat{Z}_V)$ a monoidal functor.

Proof. a) follows from the following standard geometric facts. Let $(\mathcal{F}\ell_C^{(2)})_\lambda$ be the closure of $\mathcal{F}\ell_e \times \mathcal{G}\mathfrak{r}_\lambda \times C^0 \subset \mathcal{F}\ell_C^{(2)}$ (where $e \in W$ is the unit element). Then $\mathcal{F}\ell_\lambda \times \{x_0\}$ is contained in the smooth locus of $(\mathcal{F}\ell_C^{(2)})_\lambda$, it is open in $(\mathcal{F}\ell_C^{(2)})_\lambda \times_C \{x_0\}$. It follows that Z_λ which is by definition the nearby cycles of $\delta_{\mathcal{F}\ell} \boxtimes IC_\lambda$ is constant on $\mathcal{F}\ell_\lambda$ which is open in its support (see [1, 3.3.1, Lemma 9]). Likewise, considering the preimage $(\widetilde{\mathcal{F}\ell}_C^{(2)})_\lambda$ of $(\mathcal{F}\ell_C^{(2)})_\lambda$ in $\widetilde{\mathcal{F}\ell}_C$ we see that $\widetilde{\mathcal{F}\ell}_\lambda$ is open in the support of $\mathcal{Z}_\lambda$ and the restriction of $\mathcal{Z}_\lambda$ to $\tilde{F}_\lambda$ is a free pro-unipotent local system (shifted by $\dim(\widetilde{\mathcal{F}\ell}_\lambda)$). This yields a surjection as in (a). To see existence of a canonical choice of the surjection it suffices to see that the stalk of $\hat{Z}_V$ over the point $\overline{\lambda(t)}$ has a canonical generator as a topological $\pi_1(T)$ module. This follows from the fact that the section $(1_{\widetilde{\mathcal{F}\ell}}, \lambda_{\mathcal{G}\mathfrak{r}}) : C^0 \to \widetilde{\mathcal{F}\ell}_C$ extends to C and its value at x_0 is $\lambda(t)_{\widetilde{\mathcal{F}\ell}}$.

Uniqueness of the filtration in (b) follows from the fact that $Hom^\bullet(\mathcal{J}_\lambda, \mathcal{J}_\mu) = 0$ for $\mu \not\preceq \lambda$ (Lemma 6(b)). Together with the isomorphism $\mathcal{J}_\lambda * \mathcal{J}_\mu \cong \mathcal{J}_{\lambda+\mu}$ this also implies compatibility with convolution and the monoidal property. Existence of the filtration is equivalent to the fact that $\mathcal{J}_\mu * \hat{Z}_\lambda$ admits a free-monodromic costandard filtration when μ is deep in the dominant chamber (more precisely, when $\mu + \nu$ is dominant for any weight ν of V_λ). This follows from Proposition 9 and the corresponding fact about the sheaves Z_λ established in [1]. $\square$

3.5.5. *Torus monodromy.* Every sheaf in $\mathcal{P}_{I^0,I^0}$ is monodromic with respect to $\mathbf{I} \times T$ with unipotent monodromy, since every irreducible object in $\mathcal{P}_{I^0,I^0}$ is equivariant. Thus taking logarithm of monodromy we get an action of $Sym(\mathfrak{t} \oplus \mathfrak{t})$ on $\mathcal{P}_{I^0,I^0}$ by endomorphisms of the identity functor.

Lemma 16. *a) The action of the two copies of $\mathfrak{t}$ on Δ_w, ∇_w differ by twist with the element $\bar{w} \in W_f$, where we use the notation $w \mapsto \bar{w}$ for the projection $W \to W_f$. In particular, the left action of $\mathfrak{t}$ on the objects Δ_λ, ∇_λ, $\lambda \in \Lambda$, coincides with the right one.*

b) The left action of $\mathfrak{t}\check{}$ on the objects $\mathcal{J}_\lambda$, $\lambda \in \Lambda$, $\hat{Z}_\mu$, $\mu \in \Lambda^+$ coincides with the right one.

c) The action of loop rotation monodromy on Δ_λ, ∇_λ, $\mathcal{J}_\lambda$ coincides with the image of coweight $\lambda \in \mathfrak{t}^\vee$ under the above action of $\mathfrak{t}^\vee$.

Proof. a) is clear from the definitions, the statement about $\mathcal{J}_\lambda$ in (b) follows from (a). The statement about $\hat{Z}_\lambda$ in (b) follows from the construction with nearby cycles. Part (c) is a consequence of the following observation. For a cocharacter λ let λ be the corresponding element of $\mathbf{G_F}$, and let R denote the loop rotation action of $\mathbb{G}_m$ on GK. Then we have $R(s)(\lambda) = \lambda(s)\lambda = \lambda\lambda(s)$. $\square$

4. Construction of functors

4.1. **A functor from $D^b(Coh^{G^\vee}(\tilde{\mathfrak{g}}^\vee))$.** Recall that $\hat{\tilde{\mathfrak{g}}}^\vee$ denotes the formal completion of $\tilde{\mathfrak{g}}^\vee$ at $\tilde{\mathcal{N}}$.

In this subsection we construct a monoidal functor $F : D^{G^\vee}(\hat{\tilde{\mathfrak{g}}}^\vee) \to \hat{D}$. The functor we presently construct is compatible with the equivalence $\Phi : D^b(Coh^{G^\vee}(\widehat{St})) \cong \hat{D}$ that will be established in section 9 as follows: $F \cong \Phi \circ \delta_*$, where $\delta : \hat{\tilde{\mathfrak{g}}}^\vee \to \widehat{St}$ is the embedding.

The construction is parallel to that of [1, §3], so we only recall the main ingredients of the construction referring the reader to [1] for details.

Following the strategy outlined in section 2.2, we first list compatibilities satisfied by the functor F which characterize it uniquely.

4.1.1. *Line bundles and Wakimoto sheaves.* Recall that for $\lambda \in \Lambda$ the corresponding line bundle on $\mathcal{B}$ is denoted by $\mathcal{O}_{\mathcal{B}}(\lambda)$, while $\mathcal{O}_{\hat{\tilde{\mathfrak{g}}}^\vee}(\lambda)$ is its pull-back to $\hat{\tilde{\mathfrak{g}}}^\vee$. The functor F satisfies:

$$F(\mathcal{O}_{\hat{\tilde{\mathfrak{g}}}^\vee}(\lambda) \cong \mathcal{J}_\lambda.$$

This isomorphism is compatible with the monoidal structure on the two categories, i.e. it provides a tensor isomorphism between the functor Θ (see Corollary 5) and the composition of F with the tensor functor $\lambda \mapsto \mathcal{O}_{\hat{\tilde{\mathfrak{g}}}^\vee}(\lambda)$.

4.1.2. *Twists by representations and central functors.* We have a tensor functor $Rep(G^\vee) \to Coh^{G^\vee}(\tilde{\mathfrak{g}}^\vee)$ sending a representation V to $V \otimes \mathcal{O}$. Composition of F with this functor is isomorphic to the tensor functor $V \mapsto \hat{Z}_V$ (see section 3.5).

4.1.3. *The lowest weight arrow.* We have a familiar morphism of $G^\vee$ equivariant vector bundles on $\mathcal{B}$: $\mathcal{O}_{\mathcal{B}} \otimes V_\lambda \to \mathcal{O}_{\mathcal{B}}(\lambda)$. We can pull it back to $\hat{\tilde{\mathfrak{g}}}^\vee$ to get a morphism in $Coh^{G^\vee}(\hat{\tilde{\mathfrak{g}}}^\vee)$. The functor F sends this arrow to the map ϖ_λ (notations of Proposition 15).

4.1.4. *Log monodromy endomorphism.* Notice that for $x \in G^\vee$, $\mathcal{F} \in Coh^{G^\vee}(\mathfrak{g}^\vee)$ the centralizer of x in $G^\vee$ acts on the fiber $\mathcal{F}_x$ of $\mathcal{F}$ at x. Differentiating this action one gets the action of the Lie algebra of the centralizer $\mathfrak{z}(x)$. In particular, $x \in \mathfrak{z}(x)$ produces a canonical endomorphism of $\mathcal{F}_x$, it is easy to see that it comes from a uniquely defined endomorphism of $\mathcal{F}$, which we denote by $\mathfrak{m}_{\mathcal{F}}$ (in [1] we used notation $N_{\mathcal{F}}^{taut}$). It is clear that restricting $\mathfrak{m}$ to sheaves of the form $\mathcal{F} = V \otimes \mathcal{O}_{\tilde{\mathfrak{g}}^\vee}$ one gets a tensor endomorphism of the tensor functor $V \mapsto V \otimes \mathcal{O}_{\tilde{\mathfrak{g}}^\vee}$.

We require that F sends $\mathfrak{m}_{V\otimes\mathcal{O}}$ to the monodromy endomorphism μ_V.

4.1.5. *Projection to* $\mathfrak{t}^{\check{}2}$ *and torus monodromy.* We have a canonical map $\tilde{\mathfrak{g}}\check{} \to \mathfrak{t}\check{}^*$, thus the category $D^b(Coh^{G\check{}}(\tilde{\mathfrak{g}}\check{}))$ is canonically an $\mathcal{O}(\mathfrak{t}\check{}^*)$ linear category, i.e. $\mathfrak{t}\check{}$ acts on it by endomorphisms of the identity functor. This induces a pro-nilpotent action of $\mathfrak{t}\check{}$ on $D^b(Coh^{G\check{}}(\hat{\tilde{\mathfrak{g}}}\check{}))$.

According to section 3.5.5, we have two commuting pronilpotent $\mathfrak{t}\check{}$ action on $\hat{\mathcal{P}}$ and hence on $\hat{D}$. The functor F intertwines the action of $\mathfrak{t}\check{}$ described in the previous paragraph with either of the two monodromy actions.

4.2. **Monoidal functor from sheaves on the diagonal.** We use a version of homogeneous coordinate ring construction and Serre description of the category of coherent sheaves on a projective variety.

Let $C(\tilde{\mathfrak{g}}\check{})$ the preimage of $\tilde{\mathfrak{g}}\check{} \subset \mathfrak{g}\check{} \times \mathcal{B}$ under the morphism $\mathfrak{g}\check{} \times G\check{}/U\check{} \to \mathfrak{g}\check{} \times \mathcal{B}$. Let $\overline{G\check{}/U\check{}}$ denote the affine closure of $G\check{}/U\check{}$. Notice that $G\check{}/U\check{}$ can be realized as a locally closed subscheme, namely as the orbit of a highest weight vector in the space V of a representation of G. Moreover, if the representation V is chosen appropriately, the closure of $G\check{}/U\check{}$ in V is isomorphic to $\overline{G\check{}/U\check{}}$. Define the action of the abstract Cartan $\mathfrak{t}\check{}$ on V such that $t \in \mathfrak{t}\check{}$ acts on an irreducible summand with highest weight λ by the scalar $\langle \lambda, t\rangle$. Then define a closed subscheme[4] $\overline{C} \subset \mathfrak{g}\check{} \times \mathfrak{t}\check{} \times \overline{G\check{}/U\check{}}$ by the equation $x(v) = t(v)$, $x \in \mathfrak{g}\check{}$, $t \in \mathfrak{t}\check{}$, $v \in \overline{G\check{}/U\check{}} \subset V$.

We leave the proof of the following statement to the reader

Proposition 17. *A) The scheme* $\overline{C}$ *does not depend on the choice of* V *subject to the above conditions.*

B) Consider the category of commutative rings over $\mathcal{O}(\mathfrak{t}\check{})$ *equipped with a* $G\check{}$ *action which fixes the image of* $\mathcal{O}(\mathfrak{t}\check{})$.

The following two functors on that category are canonically isomorphic:

1. $R \mapsto Hom(Spec(R), \overline{C})$ *where* Hom *stands for maps compatible with the* $G\check{}$ *action and the map to* $\mathfrak{t}\check{}$.
2. $R \mapsto \{(E_V, \iota_V) \mid V \in Rep(G\check{})\}$. *Here for* $V \in Rep(G\check{})$, $E_V \in End_R(V \otimes R)$ *and* ι_V *is a map of* R*-modules* $R \to V \otimes R$. *This data is subject to the requirements:*
   (a) *functoriality in* V
   (b) $E_{V\otimes W} = E_V \otimes Id_W + Id_V \otimes E_W$;
   (c) $\iota_{V\otimes W} = \iota_V \otimes \iota_W$;
   (d) $E_{V_\lambda} \circ \iota_V = \lambda \circ \iota_V$ *where* $\lambda \in \mathfrak{t}\check{}^* \subset \mathcal{O}(\mathfrak{t}\check{}) \to R$.

4.2.1. *Deequivariantization.* (cf. section 2.2.1(3)) We will make use of the following construction. Let $\mathcal{C}$ be an additive category linear over the field k, with an action of the tensor category $Rep(H)$ of (finite dimensional algebraic) representation of H, where H is a reductive algebraic group over k. (Recall that k is algebraically closed of characteristic zero; the definition is applicable under less restrictive assumptions).

We can then define a new category $\mathcal{C}_{deeq}$ by setting $Ob(\mathcal{C}_{deeq}) = Ob(\mathcal{C})$, $Hom_{\mathcal{C}_{deeq}}(A, B) = Hom_{Ind(\mathcal{C})}(A, \underline{\mathcal{O}}(H)(B))$, where $Ind(\mathcal{C})$ is the category of Ind objects in $\mathcal{C}$ and $\underline{\mathcal{O}}(H)$ is the object of $Ind(Rep(H))$ coming from the module of regular functions on G equipped with the action of H by left translations. Using that H is reductive over an algebraically closed field of characteristic zero we can write the Ind-object

[4]Here notations diverge from that of [1], there hat was used to denote the affine cone, while in the present paper it is used to denote completions.

$\underline{\mathcal{O}}(H)$ as $\bigoplus\limits_{\underline{V}\in IrrRep(H)} \underline{V}\otimes V^*$, where $IrrRep(H)$ is a set of representatives for isomorphism classes of irreducible H modules and for a representation $\underline{V}\in IrrRep(H)$ we let V denote the underlying vector space. Thus we have

$$Hom_{deeq}(X,Y) = \bigoplus_{\underline{V}\in IrrRep(H)} Hom(X,\underline{V}(Y))\otimes V^*.$$

For example, if $\mathcal{C} = D^b(Coh^H(X))$ where X is a scheme equipped with an H action then for $\mathcal{F},\mathcal{G}\in\mathcal{C}$ we have $Hom_{deeq}(\mathcal{F},\mathcal{G}) = Hom_{D^b(Coh(X))}(\mathcal{F},\mathcal{G})$.

When we need to make the group H explicit in the above definition we write Hom^H_{deeq} instead of Hom_{deeq}.

The category $\mathcal{C}_{deeq}$ is enriched over H-modules, i.e. every Hom space carries the structure of an H-module compatible with convolution. We refer the reader to [2] for further details and to [19] for a more general construction (cf. also [1], proof of Proposition 4).

This formalism comes in handy for deducing the following statement.

Let $Coh^{\check{G}\times\check{T}}_{fr}(\overline{C})$ be the full subcategory in $Coh^{\check{G}\times\check{T}}(\overline{C})$ consisting of objects of the form $V\otimes\mathcal{O}$, $V\in Rep(\check{G}\times\check{T})$. In other words, objects of $Coh^{\check{G}\times\check{T}}_{fr}(\overline{C})$ are representations of $\check{G}\times\check{T}$ and morphisms are given by $Hom(V_1,V_2) = Hom_{Coh^{\check{G}\times\check{T}}(\overline{C})}(V_1\otimes\mathcal{O}, V_2\otimes\mathcal{O})$. This is a tensor category under the usual tensor product of vector bundles.

Corollary 18. *Let $\mathcal{C}$ be a k-linear additive monoidal category. Suppose we are given*

1) A tensor functor $F : Rep(\check{G}\times\check{T})\to\mathcal{C}$.

2) A tensor endomorphism E of $F|_{Rep(\check{G})}$, $E_{V_1\otimes V_2} = E_{V_1}\otimes Id_{F(V_2)} + Id_{F(V_1)}\otimes E_{V_2}$.

3) An action of $\mathcal{O}(\check{\mathfrak{t}})$ on F by endomorphisms, so that for $f\in O(\check{\mathfrak{t}})$ we have $f_{V_1\otimes V_2} = f_{V_1}\otimes Id_{F(V_2)} = Id_{F(V_1)}\otimes f_{V_2}$.

4) A "lowest weight arrow" $\varpi_\lambda : F(V_\lambda)\to F(\lambda)$ making the following diagrams commutative:

$$\begin{array}{ccc}
F(V_\lambda\otimes V_\mu) & \longrightarrow & F(V_{\lambda+\mu} \\
{\scriptstyle \varpi_\lambda\otimes_{\mathcal{C}}\varpi_\mu}\big\downarrow & & \big\downarrow{\scriptstyle \varpi_{\lambda+\mu}} \\
F(\lambda)\otimes_{\mathcal{C}} F(\mu) & \xrightarrow{\sim} & F(\lambda+\mu)
\end{array}$$

$$\begin{array}{ccc}
F(V_\lambda) & \xrightarrow{\varpi_\lambda} & F(\lambda) \\
{\scriptstyle E_{V_\lambda}}\big\downarrow & & \big\downarrow{\scriptstyle \lambda} \\
F(V_\lambda) & \xrightarrow{\varpi_\lambda} & F(\lambda)
\end{array}$$

where the right vertical map is the action of the element $\lambda\in\mathfrak{t}\subset\mathcal{O}(\check{\mathfrak{t}})$ coming from (3).

Then the tensor functor F extends uniquely to a tensor functor $Coh^{\check{G}}_{fr}(C)\to\mathcal{C}$, so that E goes to the tautological endomorphism $\mathfrak{m}$ (see section 4.1.4), the action of $\mathfrak{t}$ comes from the projection $C\to\mathfrak{t}$ and the lowest weight arrow comes from the map described in section 4.1.3.

Proof. Extending the functor F to a functor $Coh^{\check{G}}_{fr}(C)\to\mathcal{C}$ is equivalent to providing a $\check{G}\times\check{T}$ equivariant homomorphism $\mathcal{O}(C)\to Hom^{\check{G}\times\check{T}}_{deeq}(1_{\mathcal{C}},1_{\mathcal{C}})$. Thus Corollary follows directly from Proposition 17. □

4.2.2. *The functor* Φ_{diag}. We now construct a monoidal functor $\Phi^{fr}_{diag} : Coh^{\check{G}\times \check{T}}_{fr}(\overline{C}) \to \hat{\mathcal{P}}$ (more precisely, a monoidal functor to $\hat{D}$ taking values in $\hat{\mathcal{P}}$).

The functor is provided by Corollary 18: we have the action of $Rep(\check{G})$ on $\hat{D}$ coming from the central functors (subsection 4.1.2), and a commuting action of $Rep(\check{T})$ coming from Wakimoto sheaves (3.3); the logarithm of monodromy endomorphisms (subsection 4.1.4) provide endomorphism E while the torus monodromy (section 3.5.5) gives an action of $\mathfrak{t} = \check{\mathfrak{t}}^*$. The morphisms described in subsection 4.1.3 yield arrows ι_V. It is not hard to check that the conditions of Corollary 18 hold.

4.3. **"Coherent" description of the anti-spherical (generalized Whittaker) category.** Consider the composition $Ho(Coh^{\check{G}\times \check{T}}_{fr}(\overline{C})) \to Ho(\hat{\mathcal{P}}) \to \hat{D}$ where Ho denotes the homotopy category of complexes of objects in the given additive category and the first arrow is induced by Φ^{fr}_{diag}; this composition will be denoted by Φ^{Ho}_{diag}.

Let $Acycl \subset Ho(Coh^{\check{G}\times \check{T}}_{fr}(\overline{C}))$ be the subcategory of complexes whose restriction to the open subscheme C is acyclic.

Proposition 19. $\Phi^{Ho}_{diag} : Acycl \to 0$.

Proof. Proposition follows from existence of a filtration on $\hat{Z}_\lambda$ with associated graded being the sum of Wakimoto sheaves (Proposition 15(b)) by an argument parallel to [1, 3.7]. □

We have $D^b(Coh^{\check{G}}(\tilde{\check{\mathfrak{g}}})) \cong Ho(Coh^{\check{G}\times \check{T}}_{fr}(\overline{C}))/Acycl$, thus the Proposition yields a functor $D^b(Coh^{\check{G}}(\tilde{\check{\mathfrak{g}}})) \to \hat{D}$. The log monodromy action of $\check{\mathfrak{t}}$ on the identity functor of $\hat{D}$ is pro-unipotent, it is easy to deduce that the functor factors canonically through a functor $D^b(Coh^{\check{G}}(\widehat{\tilde{\check{\mathfrak{g}}}})) \to \hat{D}$, we denote the latter functor by Φ_{diag}.

A closely related functor $F : D^b(Coh^{\check{G}}(\tilde{\mathcal{N}})) \to D_{I,I}$ was constructed in [1, §3].

Lemma 20. *Let* $i : \tilde{\mathcal{N}} \to \widehat{\tilde{\check{\mathfrak{g}}}}$ *be the embedding. The following diagrams commute up to a natural isomorphism:*

$$\begin{tikzcd} D^b(Coh^{\check{G}}(\widehat{\tilde{\check{\mathfrak{g}}}})) \arrow[r, "\Phi_{diag}"] \arrow[d, "i^*"'] & \hat{D} \arrow[d, "\pi_*"] \\ D^b(Coh^{\check{G}}(\tilde{\mathcal{N}}) \arrow[r, "Res^I_{I^0}\circ F"] & D_{I^0,I} \end{tikzcd}$$

$$\begin{tikzcd} D^b(Coh^{\check{G}}(\tilde{\mathcal{N}})) \arrow[r, "F"] \arrow[d, "i_*"'] & D_{I,I} \arrow[d, "Res^I_{I^0}\circ\pi^*[r]"] \\ D^b(Coh^{\check{G}}(\widehat{\tilde{\check{\mathfrak{g}}}}) \arrow[r, "\Phi_{diag}"] & \hat{D} \end{tikzcd}$$

where Res stands for restriction of equivariance, and $r = \mathrm{rank}(\mathfrak{g})$.

Proof. To check commutativity of the first diagram it suffices to prove the similar commutativity for functors on the categories of finite complexes in $Coh^{\check{G}}_{fr}(\widehat{\tilde{\check{\mathfrak{g}}}})$. This follows from the isomorphisms $\pi_*(\hat{Z}_V) \cong Z_V$ (Proposition 13(d)) $\pi_*(\mathcal{J}_\lambda) \cong J_\lambda$ (Lemma 6(c)) which are easily seen to be compatible with monodromy endomorphism and lowest weight arrows.

Now commutativity of the second diagram follows from the fact that $\pi^*\pi_* : \hat{D} \to \hat{D}$ is isomorphic to the functor of tensoring with Koszul complex over the algebra $U(\mathfrak{t})$ composed with shift by $-r$, $r = \dim(T)$, see Lemma 45(b). $\square$

4.3.1. *Equivalence Φ_{IW}.* We are now ready to establish (6).

Proposition 21. *$Av_{IW} \circ \Phi_{diag}$ is an equivalence.*

Proof. We first show that Φ_{diag} is fully faithful. It suffices to show that

$$Hom(\mathcal{F}, \mathcal{G}) \tilde{\longrightarrow} Hom(Av_{IW}\Phi_{diag}(\mathcal{F}), Av_{IW}\Phi_{diag}(\mathcal{G}))$$

when $\mathcal{F} = i_*(\mathcal{F}')$, $\mathcal{F}' \in D^b(Coh^{G\check{}}(\tilde{\mathcal{N}}))$, then the statement follows since the image of i_* generates $D^b(Coh^{G\check{}}_{\tilde{\mathcal{N}}}(\tilde{\mathfrak{g}}\check{}))$ under extensions, so we get the isomorphism for $\mathcal{F}$, $\mathcal{G} \in D^b(Coh_{\tilde{\mathcal{N}}}(\tilde{\mathfrak{g}}\check{}))$. Passing to the limit we then get the isomorphism for all $\mathcal{F}$, $\mathcal{G} \in D^b(Coh^{G\check{}}(\widehat{\tilde{\mathfrak{g}}\check{}}))$.

Using the parallel statement in the non-monodromic setting proved in [1, §4] and the first commutative diagram in Lemma 20 (or rather the statement obtained from it by left-right swap) we get:

$$\begin{aligned} Hom(i_*(\mathcal{F}'), \mathcal{G}) &\cong Hom(\mathcal{F}', i^*\mathcal{G}[-r]) \cong Hom_{D^I_{IW}}(Av_{IW}F(\mathcal{F}')), Av_{IW}F(i^*\mathcal{G})[-r]) \cong \\ &Hom_{D^I_{IW}}\left(Av_{IW}F(\mathcal{F}'), (Av^I_{I^0})^{left}Av_{IW}(\Phi_{diag}(\mathcal{G}))[-r]\right) \cong \\ &Hom_{D^{I^0}_{IW}}\left((Res^I_{I^0})^{left}Av_{IW}F(\mathcal{F}')), Av_{IW}(\Phi_{diag}(\mathcal{G}))[-r]\right) \cong \\ &\qquad Hom\left(Av_{IW}\Phi_{diag}(\mathcal{F}), Av_{IW}\Phi_{diag}(\mathcal{G}')\right), \end{aligned}$$

where we used that $i^*[-r]$ is *right* adjoint to i_*.

This shows that the functor is fully faithful. Again using the parallel statement in the non-monodromic setting and Lemma 20 we see that the essential image of Φ_{diag} contains the image of the functor of restricting the equivariance $D^I_{IW} \to D_{I^0,IW}$. Since any object in $D^b(Coh^{G\check{}}(\widehat{\tilde{\mathfrak{g}}\check{}}))$ is an inverse limit of objects in $D^b(Coh^{G\check{}}_{\tilde{\mathcal{N}}}(\tilde{\mathfrak{g}}\check{}))$ whose image under $i^* : Pro(D^b(Coh^{G\check{}}_{\tilde{\mathcal{N}}}(\tilde{\mathfrak{g}}\check{}))) \to Pro(D^b(Coh^{G\check{}}(\tilde{\mathcal{N}})))$ lies in $D^b(Coh^{G\check{}}(\tilde{\mathcal{N}}))$, the corresponding pro-object in $D^{I^0}_{IW}$ lies by definition in $\hat{D}_{IW}$, which implies essential surjectivity. $\square$

4.4. **$\hat{D}$ is a category over $St/G\check{}$, $D_{I^0,I}$ is a category over $St'/G\check{}$.** The goal of this section is to construct an action of the tensor category $D^{G\check{}}_{perf}(\widehat{St})$ on $\hat{D}$ and of $D^{G\check{}}_{perf}(St')$ on $D_{I^0,I}$, both categories are equipped with the tensor structure coming from tensor product of perfect complexes.

4.4.1. *The action of the tensor categories $Coh^{G\check{}}_{free}(C_{St})$, $Coh^{G\check{}}_{free}(C_{St'})$.* We let $\overline{C}_{St}$ be the preimage of diagonal under the map $\overline{C}_{\tilde{\mathfrak{g}}\check{}} \times \overline{C}_{\tilde{\mathfrak{g}}\check{}} \to \mathfrak{g}\check{} \times \mathfrak{g}\check{}$, and let $\overline{C}_{St'}$ be the preimage of 0 under the second projection to $\mathfrak{t}\check{}$. We have open subsets $C_{St} \subset \overline{C}_{St}$ and $C_{St'} \subset \overline{C}_{St'}$ where the action of $T\check{} \times T\check{}$ is free and $St = C(St)/T\check{}^2$, $St' = C(St')/T\check{}^2$.

We apply Corollary 18 in the following setting: the group $G\check{}$ is replaced by $G\check{}^2$ and $\mathcal{C}$ is the category of functors $\hat{D} \to \hat{D}$ (respectively $D_{I^0,I} \to D_{I^0,I}$).

We have two actions of $Rep(T\check{})$ coming from, respectively, left and right convolution with Wakimoto sheaves. We consider the action of $Rep(G\check{}^2)$ obtained as composition of restriction to the diagonal copy of $G\check{}$ and the action by central functors. The nearby cycles monodromy acting on the cental functor defines a tensor

endomorphism E of the $\check{G}^2$ action, while the torus monodromy defines an action of $\check{\mathfrak{t}}^2$. It is not hard to see that conditions of Corollary 18 are satisfied. Thus we get an action of $Coh_{fr}^{\check{G}^2\times\check{T}^2}(\overline{C}^2)$ on $\hat{D}$, $D_{I^0,I}$.

The fact that the action of $\check{G}^2$ factors through restriction to diagonal is easily seen to imply that the action factors canonically through a uniquely defined action of $Coh^{\check{G}\times\check{T}^2}(\overline{C}^2)$. Furthermore, since the isomorphism between the two actions of $\check{G}$ is compatible with the tensor endomorphism E, both actions factor through a uniquely defined action of $Coh_{fr}^{\check{G}\times\check{T}^2}(\overline{C}_{St})$. Finally, since the second (right monodromy) action of $\check{\mathfrak{t}}$ on $D_{I^0,I}$ vanishes, the action of $Coh_{fr}^{\check{G}\times\check{T}^2}(\overline{C}_{St})$ factors through $Coh_{fr}^{\check{G}\times\check{T}^2}(\overline{C}_{St'})$. We denote the two actions by Φ_{fr}, Φ'_{fr} respectively.

4.4.2. *Extending the actions to the perfect derived categories.* Our next goal is to extend the action described in the previous subsection to complexes. We encounter the standard non-functoriality of cone issue, which we circumvent in the following way.

We use the equivalences $Ho(\hat{\mathcal{T}})\widetilde{\longrightarrow}\hat{D}$, $Ho(\mathcal{T})\widetilde{\longrightarrow}D_{I^0,I}$.

Assume given a finite complex $\mathcal{F}^\bullet$ of objects in $Coh^{\check{G}\times\check{T}^2}(\overline{C}_{St})$, where each term $\mathcal{F}^i$ is a trivial vector bundle twisted by a representation U^i of $\check{G}\times\check{T}^2$. Pick λ, $\mu\in\Lambda$ so that for each character (λ_j,μ_j) of $\check{T}^2$ appearing in one of the representations U^i we have $\lambda+\lambda_j\in(-\Lambda^+)$, $\mu+\mu_j\in\Lambda^+$.

In view of Corollary 12 and Lemma 4(d) the functor $\Phi_{fr}\circ\mathcal{J}^l_\lambda\circ\mathcal{J}^r_\mu$ sends $\hat{\mathcal{T}}$ to $\hat{\mathcal{P}}$, where $\mathcal{J}^l_\lambda: X\mapsto\mathcal{J}_\lambda * X$, $\mathcal{J}^r_\mu: X\mapsto X*\mathcal{J}_\mu$.

We now define a functor $\hat{D}\to\hat{D}$ as the composition:

$$\hat{D}\xrightarrow{\mathcal{J}^l_{-\lambda}\circ\mathcal{J}^r_{-\mu}}\hat{D}\xleftarrow{\ \sim\ } Ho(\hat{\mathcal{T}})\xrightarrow{\Phi_{fr}(\mathcal{F}^\bullet)\circ\mathcal{J}^l_\lambda\circ\mathcal{J}^r_\mu} Ho(\hat{\mathcal{P}})\to\hat{D}.$$

We claim that different choices of λ, μ produce canonically isomorphic functors. This follows from existence of a canonical up to homotopy quasi-isomorphism $\mathcal{J}_{-\lambda}*T\to T'$, $T'\to T*\mathcal{J}_\mu$, where T, T' are finite complexes of objects in $\hat{\mathcal{T}}$ and T' represents the object in the derived category corresponding to $\mathcal{J}_{-\lambda}*T$ (respectively, $T*\mathcal{J}_\mu$), $\lambda,\mu\in\Lambda^+$.

Thus we get a well defined functor $Ho(Coh_{fr}^{\check{G}\times\check{T}^2}(\overline{C}_{St}))\to End(\hat{D})$. It is not hard to see from the definition that the last arrow carries a natural monoidal structure.

Let $Acycl_{St}\subset Ho(Coh_{fr}^{\check{G}\times\check{T}^2}(\overline{C}_{St})$ be the subcategory of complexes whose restriction to C_{St} is acyclic. As in Proposition 19, the fact that the lowest weight arrow ϖ_λ extends to a filtration by Wakimoto sheaves compatible with convolution implies that $Acycl$ acts on $\hat{D}$ by zero. Thus we obtain an action of $Ho(Coh_{fr}^{\check{G}\times\check{T}^2}(\overline{C}_{St}))/Acycl\cong D^{\check{G}}_{perf}(St)$. Finally, since the action of the monodromy endomorphism is pro-unipotent, we conclude that the action factors through $D^{\check{G}}_{perf}(\widehat{St})$.

A parallel argument (with the previous sentence omitted) endows $D_{I^0,I}$ with an action of $D^{\check{G}}_{perf}(St')$.

4.4.3. *Compatibility between the two actions.* For future reference we record a compatibility between the two actions.

Lemma 22. *For $\mathcal{F} \in D^{G\check{}}_{perf}(\widehat{St})$, $X \in \hat{D}$ and $Y \in D$ we have canonical isomorphisms*

$$\pi_*(\mathcal{F}(X)) \cong i^*_{St}(\mathcal{F})(\pi_*(X)),$$
$$\pi^*(i^*_{St}(\mathcal{F})(Y)) \cong \mathcal{F}(\pi^*(Y)),$$

where i_{St} denotes the closed embedding $St' \to \widehat{St}$. The isomorphism is functorial in $\mathcal{F}$, X, Y and it is compatible with the monoidal structure of the action functor.

Proof. Comparing the procedures of extending the action to the category of complexes for $\hat{D}$ and D and using that π_* sends $\hat{\mathcal{T}}$ into $\mathcal{T}$ we see that to get the first isomorphism it suffices to construct a functorial isomorphism for $\mathcal{F} \in Coh^{G\check{}}_{fr}(\widehat{St})$. This follows from $\pi_*(\hat{Z}_V) = Z_V$, $\pi_*(\mathcal{J}_\lambda) = J_\lambda$, where the second isomorphism is compatible with the log monodromy endomorphism and the last two isomorphisms are compatible with the lowest weight arrows. The second isomorphism can be deduced using the adjunction

$$Hom(\mathcal{F}(X), X') \cong Hom(X, \mathcal{F}^*(X'))$$

which holds for both actions; here $\mathcal{F}^* = R\underline{Hom}(\mathcal{F}, \mathcal{O})$. In view of the isomorphism $i^*(\mathcal{F}^*) \cong (i^*(\mathcal{F}))^*$ the second isomorphism follows from the first one. □

5. The anti-spherical projector.

Set $\hat{\Xi} = \hat{T}_{w_0}$, $\Xi = T_{w_0}$.

Recall that $D^{I_0}_{IW}$ is the derived category of Iwahori-Whittaker sheaves on $\widetilde{\mathcal{F}\ell}$. We have *averaging functors* $Av_{I^0_-,\psi} : D_{I_0,I_0} \to D^{I_0}_{IW}$ and $Av_{I^0} : D^{I_0}_{IW} \to D_{I^0,I^0}$.

5.1. $\hat{\Xi}$ and Whittaker averaging.

Proposition 23. *(see* [13, 4.6]*) a) Right convolution with $\hat{\Xi}$ is isomorphic to $Av_{I^0} \circ Av_{I^0_-,\psi}$.*

b) Convolution with $\hat{\Xi}$ is isomorphic to its left and right adjoint.

c) The full subcategory in $\hat{\mathcal{T}}$ consisting of direct sums of copies of $\hat{\Xi}$ is a monoidal subcategory. It is equivalent to $Coh_{free}(\widehat{\mathfrak{h}^* \times_{\mathfrak{h}^*/W_f} \mathfrak{h}^*})$, where "hat" stands for completion at zero. □

5.2. Tilting property of $\hat{\Xi} * Z_\lambda$.

Proposition 24. *For $V \in Rep(G\check{})$ we have*

a) $\Xi * Z_V \in \mathcal{T}$.

b) $\hat{\Xi} * \mathcal{Z}_V \in \hat{\mathcal{T}}$.

Proof. a) is proven in [1]. Part (b) follows from Lemma 9, compatibility of central functors with direct image and Propostion 23(a) and part (a) of this Proposition. □

Corollary 25. *For $T \in \mathcal{T}$, $\hat{T} \in \hat{\mathcal{T}}$ we have*

$$Ext^{\neq 0}(\mathcal{J}_\lambda * \Xi * J_\mu * Z_\nu, T) = 0,$$

$$Ext^{\neq 0}(\mathcal{J}_\lambda * \hat{\Xi} * \mathcal{J}_\mu * \hat{Z}_\nu, \hat{T}) = 0$$

for $\lambda, \mu \preceq 0$.

Proof. We have

$$Ext^\bullet(\mathcal{J}_\lambda * \hat{\Xi} * \mathcal{J}_\mu * Z_\nu, T) \cong Ext^\bullet(\mathcal{J}_\lambda * \Xi * \mathcal{J}_\mu * Z_\nu, T * \mathcal{J}_{-\mu}).$$

Comparing Proposition 24 with Corollary 12 we see that $\mathcal{J}_\lambda * \Xi * \hat{Z}_\nu$ admits a free monodromic standard filtration, while $T * \mathcal{J}_{-\mu}$ admits a free-monodromic co-standard filtration, which implies the second vanishing. The first one is similar. □

5.3. **Convolution with $\hat{\Xi}$ and the Springer map.** Recall that p_{Spr} denotes the projection $\tilde{\mathfrak{g}}\check{} \to \mathfrak{g}\check{}$.

Proposition 26. *The equivalence $D_{IW} \cong D^b(Coh^{G\check{}}_{\mathcal{N}}(\tilde{\mathfrak{g}}\check{}))$ intertwines the endo-functor $\mathcal{F} \mapsto \hat{\Xi} * \mathcal{F}$ with the endo-functor $p^*_{Spr} p_{Spr*}$.*

We start with a

Lemma 27. *Recall that $\Phi^{I^0}_{IW}$ denotes the equivalence $D^b(Coh^{G\check{}}(\widehat{\tilde{\mathfrak{g}}\check{}})) \cong \hat{D}_{IW}$.*

a) The object $\Phi^{-1}_{IW}(Av_{IW}(\hat{\Xi})) \in \hat{\mathcal{P}}_{IW}$ is canonically isomorphic to $\mathcal{O}(\mathfrak{h}) \otimes_{\mathcal{O}(\mathfrak{h})^W} \mathcal{O}$.

b) The composed functor $\mathcal{F} \mapsto \hat{\Xi} * (\Phi^{I^0}_{IW} \circ p^*_{Spr}(\mathcal{F}))$ is isomorphic to the functor $\mathcal{F} \mapsto \Phi_{IW}(\mathcal{O}(\mathfrak{h}) \otimes_{\mathcal{O}(\mathfrak{h}/W)} p^*_{Spr})$.

c) For $\mathcal{F} \in \hat{D}_{IW}$ we have $\hat{\Xi} * \mathcal{F} = 0$ iff $(\Phi^{I^0}_{IW})^{-1}(\mathcal{F}) \in Ker(p_{Spr*})$.

Proof of the Lemma. a) To show that $(\Phi^{I^0}_{IW})^{-1}(\hat{\Xi})$ is noncanonically isomorphic to $\mathcal{O}^{|W|}$ we use the filtration on $\hat{\Xi}$ with associated graded being the sum of ∇_w, $w \in W_f$. Since $\nabla_e \cong \Phi_{diag}(\mathcal{O})$ and $Av_{IW}(\nabla_w) \cong Av_{IW}(\nabla_{w'})$ when $w' \in wW_f$, it follows that the image of $Av_{IW}(\hat{\Xi})$ under $(\Phi^{I^0}_{IW})^{-1}$ admits a filtration of length $|W|$ where each subquotient is isomorphic to $\mathcal{O}$. It is well known that $Ext^1_{Coh^{G\check{}}(\tilde{\mathfrak{g}}\check{})}(\mathcal{O}, \mathcal{O}) = 0$, hence we get an isomorphism. A canonical isomorphism now follows from Proposition 23(c), which yields an isomorphism $\mathcal{O}(\mathfrak{h}) \otimes_{\mathcal{O}(\mathfrak{h}/W)} \mathcal{O}^{|W|} \cong \hat{\Xi} * \hat{\Xi}$.

b) The functor $\Phi^{I^0}_{IW} \circ p^*_{Spr} : D^b(Coh^{G\check{}}(\widehat{\tilde{\mathfrak{g}}\check{}})) \to \hat{D}_{IW}$ comes from the *central* action of $D^b(Coh^{G\check{}}(\widehat{\tilde{\mathfrak{g}}\check{}}))$ on $\hat{D}$. Since this action commutes with the functor of convolution with $\hat{\Xi}$, (b) follows from (a).

c) The kernel of p_{Spr*} is the (right) orthogonal to the objects $\mathcal{O} \otimes V$, $V \in Rep(G\check{})$. So we need to show that $\hat{\Xi} * \mathcal{F} = 0 \iff Hom_{D_{IW}}(Av_{IW}(\hat{Z}_\lambda), \mathcal{F}) = 0$ for all $\lambda \in \Lambda^+$. First, if $\hat{\Xi} * \mathcal{F} = 0$ then by self-adjointness of convolution with $\hat{\Xi}$, $Hom(\hat{\Xi} * Av_{IW}(\hat{Z}_\lambda), \mathcal{F}) = 0$. We have $\hat{\Xi} * \hat{Z}_\lambda \cong \hat{Z}_\lambda * \hat{\Xi}$ and $Av_{IW}(\hat{Z}_\lambda * \hat{\Xi})$ admits a filtration where each subquotient is isomorphic to $Av_{IW}(\hat{Z}_\lambda)$. By a standard argument it follows that $Hom_{\hat{D}_{IW}}(Av_{IW}(\hat{Z}_\lambda), \mathcal{F}) = 0$. Conversely, suppose that $\hat{\Xi} * \mathcal{F} \neq 0$. We need to show that $Hom_{\hat{D}_{IW}}(Av_{IW}(\hat{Z}_\lambda), \mathcal{F}) \neq 0$ for some λ. Without loss of generality we can assume that $\mathcal{F} \in \hat{\mathcal{P}}_{IW}$ (recall that convolution with $\hat{\Xi}$ is exact). Then, since $Hom_{\hat{D}_{IW}}(\nabla^{IW}_w, \hat{\Xi} * \mathcal{F})$ depends only on the 2-sided coset $W_f w W_f$, we see that $Hom_{\hat{D}_{IW}}(\nabla_w, \hat{\Xi} * \mathcal{F}) \neq 0$ for some w which is maximal in its 2-sided W_f-coset. Using the tilting property of $\hat{\Xi} * \hat{Z}_\lambda$ one sees that for such w the object ∇^{IW}_w is a quotient of $\hat{\Xi} * Av_{IW}(\hat{Z}_\lambda)$ if $\lambda \in W_f w W_f$. Thus $Hom(\hat{\Xi} * Av_{IW}(\hat{Z}_\lambda), \Xi * \mathcal{F}) \neq 0$, hence $Hom(Av_{IW}(\hat{Z}_\lambda * \Xi), \mathcal{F}) \neq 0$ and $Hom(Av_{IW}(\hat{Z}_\lambda), \mathcal{F}) \neq 0$.

Proof of Proposition 26. Lemma 27(c) yields a morphism $p^*_{Spr} \circ F_{\hat{\Xi}} \to p^*_{Spr}$ where $F_{\hat{\Xi}} : \mathcal{F} \to \Phi^{-1}_{IW}(\hat{\Xi} * \Phi_{IW}(\mathcal{F}))$. By adjointness we get a morphism $F_{\hat{\Xi}} \to$

$p_{Spr*}p^*_{Spr}$. This map is an isomorphism on the image of p^*_{Spr} by Lemma 27 (b) and on $Ker(p_{Spr*})$ by Lemma 27(c). The two subcategories together generate the category in the weak sense (their common orthogonal is zero), since both functors are self-adjoint, it follows that the arrow is an isomorphism. □

6. Properties of Φ_{perf}

Recall the actions defined in subsection 4.4.2 and objects $\hat{\Xi} = \widehat{T_{w_0}}$, $\Xi = T_{w_0}$. We define $\widehat{\Phi}_{perf} : D^b_{perf}(Coh^{G\check{}}(St)) \to \hat{D}$, $\widehat{\Phi}_{perf}(\mathcal{F}) = \mathcal{F}(\hat{\Xi})$ and
$\Phi_{perf} : D^b_{perf}(Coh^{G\check{}}(St)) \to D_{I_0,I}$, $\Phi_{perf}(\mathcal{F}) = \mathcal{F}(\Xi)$.

6.1. **Compatibility of Φ_{perf} with projection $St \to \tilde{\mathfrak{g}}\check{}$.** We start by recording some of the compatibilities following directly from the definitions.

Lemma 28. *The following diagrams commute up to a natural isomorphism:*

$$\begin{array}{ccc} D^b(Coh^{G\check{}}(\widehat{St})) & \xrightarrow{i^*} & D^b(Coh^{G\check{}}(St')) \\ \widehat{\Phi}_{perf}\Big\downarrow & & \Big\downarrow\Phi_{perf} \\ \hat{D} & \xrightarrow{\pi_*} & D \end{array}$$

$$\begin{array}{ccc} D^b(Coh^{G\check{}}(\widehat{\tilde{\mathfrak{g}}\check{}})) & \xrightarrow{pr^*_{Spr_1}} & D^b(Coh^{G\check{}}(\widehat{St})) \\ \hat{\Phi}_{IW}\Big\downarrow & & \Big\downarrow\hat{\Phi}_{perf} \\ \hat{D}_{IW} & \xrightarrow{Av^{right}_{I^0}} & \hat{D} \end{array}$$

$$\begin{array}{ccc} D^b(Coh^{G\check{}}(\widehat{\tilde{\mathfrak{g}}\check{}})) & \xrightarrow{(pr'_{Spr_1})^*} & D^b(Coh^{G\check{}}(St')) \\ \hat{\Phi}_{IW}\Big\downarrow & & \Big\downarrow\Phi_{perf} \\ \hat{D}_{IW} & \xrightarrow{\pi_*\circ Av^{right}_{I^0}} & \hat{D} \end{array}$$

Proof. Commutativity of the first diagram follows from the corresponding compatibility for action (Lemma 22) and the isomorphism $\pi_*(\hat{\Xi}) \cong \Xi$ (Proposition 11(b)). To see commutativity of the second one observe that the functor $Av^{right}_{I^0}$ of averaging with respect to the right action of I^0 commutes with convolution on the left. For $\mathcal{F} \in D^b(Coh^{G\check{}}(\widehat{\tilde{\mathfrak{g}}\check{}}))$ the object $pr^*_{Spr,1}(\mathcal{F}) \in D^{G\check{}}_{perf}(\widehat{St})$ acts by left convolution with $\Phi_{diag}(\mathcal{F})$, thus the required commutativity follows from Proposition 23(a). The proof for the third diagram is similar to the second one. □

The main goal of this section is the following

Proposition 29. *The functor $D^{G\check{}}_{perf}(\widehat{St}) \times \hat{D}_{IW} \to \hat{D}_{IW}$ sending $(\mathcal{F}, \mathcal{G})$ to $\widehat{\Phi}_{perf}(\mathcal{F}) *$ $\mathcal{G}$ extends to an action of the monoidal category $D^b(Coh^{G\check{}}(St))$ on $\hat{D}_{IW}$, so that the equivalence Φ_{IW} is compatible with the structure of a module category over the monoidal category $D^{G\check{}}_{perf}(\widehat{St})$.*

It suffices to construct an isomorphism

$$\widehat{\Phi}_{IW}(\mathcal{F} * \mathcal{G}) \cong \widehat{\Phi}_{perf}(\mathcal{F}) * \Phi_{IW}(\mathcal{G}) \tag{11}$$

functorial in $\mathcal{F}$, $\mathcal{G}$.

When $\mathcal{F} \cong \mathcal{O}$, so that $\widehat{\Phi}(\mathcal{F}) \cong \hat{\Xi}$, the isomorphism (for any $\mathcal{G}$) is provided by Proposition 26. Since the functors commute with twist either by a line bundle or by a representation of $G\check{}$ we get an isomorphism for $\mathcal{F}$ of the form $\mathcal{O}(\lambda, \mu) \otimes V$, $V \in Rep(G\check{})$, this isomorphism is functorial in $\mathcal{F}$, $\mathcal{G}$.

By a standard argument, any object in $D^{G\check{}}_{perf}(St)$ is a direct summand in one represented by a finite complex of sheaves of the form $\mathcal{O}(\lambda_i, \mu_i) \otimes V_i$, where λ_i, $\mu_i \preceq 0$, thus we can assume without loss of generality that $\mathcal{F}$ is of this form. Pick $\nu \in \Lambda^+$ such that $\mu_i + \nu \in \Lambda^+$ for al i. We can choose a finite complex of free-monodromic tilting objects in $\hat{\mathcal{P}}_{IW}$ representing $\mathcal{J}_{-\nu} * \mathcal{G}$, then $\mathcal{G}$ is represented by a finite complex of objects $\mathcal{J}_\nu * \hat{T}_j$, $\hat{T}_j \in \hat{\mathcal{T}}_{IW}$.

We claim that

$$(\mathcal{J}_{\lambda_i} * \hat{\Xi} * \mathcal{J}_{\mu_i}) * (\mathcal{J}_{\nu_j} * \hat{T}_j) \in \hat{\mathcal{P}}_{IW},$$
$$\Phi_{IW}^{-1}(\mathcal{J}_{\lambda_i} * \hat{\Xi} * \mathcal{J}_{\mu_i}) * (\mathcal{J}_{\nu_j} * \hat{T}_j) \in Coh^{G\check{}}(\widehat{\tilde{\mathfrak{g}}\check{}}).$$

Here the first claim follows from Lemma 4(d) the second one follows from Lemma 30(a) below.

It is easy to check that on the coherent side the convolution of corresponding objects in the derived category is presented by the bi-complex of convolutions: convolution is the composition of pull-back and push-forward, our object are acyclic for pull-back, and the assumption on μ_i ensures the pairwise pull-back is adjusted to push-forward.

Also the convolution of the objects in the derived categories of perverse sheaves represented by the two complexes is represented by the bicomplex of convolutions, this follows from [13]. The Proposition follows. □

Lemma 30. *a)* $(\widehat{\Phi}^{I^0}_{IW})^{-1}(\Xi * \mathcal{J}_\mu * \hat{T}) \in Coh^{G\check{}}(\widehat{\tilde{\mathfrak{g}}\check{}})$ *for* $\mu \in \Lambda^+$, $\hat{T} \in \hat{\mathcal{T}}_{IW}$.

b) $(\Phi^I_{IW})^{-1}(j^{IW}_{w*}) \in Coh^{G\check{}}(\tilde{\mathcal{N}})$ *for any* $w \in W/W_f$.

c) $(\widehat{\Phi}^{I^0}_{IW})^{-1}(\nabla^{IW}_w) \in Coh^{G\check{}}(\widehat{\tilde{\mathfrak{g}}\check{}})$ *for any* $w \in W/W_f$.

Proof. a) An object in $\mathcal{F} \in D^b(Coh^{G\check{}}(\widehat{\tilde{\mathfrak{g}}\check{}}))$ lies in the abelian heart iff for large λ we have $R^i\Gamma(\mathcal{F} \otimes \mathcal{O}(\lambda)) = 0$ for $i \neq 0$. Since $R^i\Gamma(\mathcal{F}) = Hom^{G\check{}}_{deeq}(\mathcal{O}, \mathcal{F})$, it suffices to show that $Hom^i_{\hat{\mathcal{P}}_{IW}}(\hat{\Xi} * Av_{IW}(\hat{Z}_\lambda), J_\mu * \hat{T}) = 0$ for $i \neq 0$. This follows from Proposition 24(b) and the fact that $J_\mu * \hat{T}$ has a free-monodromic costandard filtration by Lemma 4(d).

Similarly, the first statement in (b) follows from $Ext^i_{D^I_{IW}}(Av_{IW}(J_{-\lambda} * Z_\mu), j^{IW}_{w*}) = 0$ for $i \neq 0$, $\lambda \in \Lambda^+$. The latter Ext vanishing is clear from the fact that $Av_{IW}(Z_\mu)$ is tilting in $\mathcal{P}^I_{IW}$ [1, Theorem 7], hence $Av_{IW}(J_{-\lambda} * Z_\mu)$ admits a costandard filtration. The proof of (c) is parallel to that of (b), with (co)standard replaced by free monodromic (co)standard. □

6.2. Φ_{perf} **is a full embedding.** It suffices to show that the map

$$Hom^\bullet(V \otimes \mathcal{O}_{St}(\lambda, \mu), V' \otimes \mathcal{O}_{St}(\lambda', \mu')) \to Hom^\bullet(\hat{Z}_V * \mathcal{J}_\lambda * \hat{\Xi} * \mathcal{J}_\mu, \hat{Z}_{V'} * \mathcal{J}_{\lambda'} * \hat{\Xi} * \mathcal{J}_{\mu'})$$

induced by Φ is an isomorphism.

The functor Φ sends twisting by a line bundle to convolution by Wakimoto sheaves, and twisting by a representation of $G\check{}$ to the central functor. Since adjoint to such a twist is twist by the dual representation, and similar adjunction holds for the central functors and convolution by Wakimoto sheaves, we see that it suffices to consider the case when $\lambda = 0 = \mu'$ and V is trivial.

Then we have:

$$Hom_{\hat{D}}(\hat{\Xi} * \mathcal{J}_\mu, \mathcal{J}_{\lambda'} * \hat{Z}_{V'} * \hat{\Xi}) \cong Hom_{\hat{D}_{IW}}(\hat{\Xi} * \mathcal{J}_\mu, \mathcal{J}_{\lambda'} * \hat{Z}_{V'}) \cong$$
$$Hom_{D^b(Coh^{G^\check{}}(\tilde{\mathfrak{g}}^\check{}))}(p^*_{Spr}p_{Spr*}(\mathcal{O}_{\tilde{\mathfrak{g}}^\check{}}(\mu)), \mathcal{O}_{\tilde{\mathfrak{g}}^\check{}}(\lambda')\otimes V') \cong Hom_{D^b(Coh^{G^\check{}}(\tilde{\mathfrak{g}}^\check{}))}(p_{Spr,2*}p^*_{Spr,1}(\mathcal{O}_{\widehat{\tilde{\mathfrak{g}}^\check{}}}(\mu)), \mathcal{O}_{\widehat{\tilde{\mathfrak{g}}^\check{}}}(\lambda') \otimes V').$$

Here the first isomorphism comes from the fact that right convolution with $\hat{\Xi}$ is isomorphic to $Av_{I^0} \circ Av_{IW}$. The second isomorphism uses the "coherent" description of the Iwahori-Whittaker category along with the fact that left convolution with $\hat{\Xi}$ corresponds to $p^*_{Spr}p_{Spr*}$ on the coherent side. Finally, the last isomorphism comes from: $p^*_{Spr}p_{Spr*} \cong p_{Spr,2*}p^*_{Spr,1}$, which follows from base change for coherent sheaves and the fact that $Tor^{\mathcal{O}(\mathfrak{g})}_{>0}(\mathcal{O}_{\tilde{\mathfrak{g}}^\check{}}, \mathcal{O}_{\tilde{\mathfrak{g}}^\check{}}) = 0$.

Using adjointness we get:

$$Hom_{D^b(Coh^{G^\check{}}(\tilde{\mathfrak{g}}^\check{}))}(pr_{2*}pr_1^*(\mathcal{O}_{\tilde{\mathfrak{g}}^\check{}}(\mu)), \mathcal{O}_{\tilde{\mathfrak{g}}^\check{}}(\lambda')\otimes V') \cong Hom_{D^b(Coh(St))}(pr_1^*(\mathcal{O}_{\tilde{\mathfrak{g}}^\check{}}(\mu)), pr_2^*(\mathcal{O}_{\tilde{\mathfrak{g}}^\check{}}(\lambda') \otimes V')).$$

Since $\Phi : pr_1^*(\mathcal{O}_{\tilde{\mathfrak{g}}^\check{}}(\mu)) \mapsto \mathcal{J}_\mu * \hat{\Xi}$, $\Phi : \mathcal{O}_{\tilde{\mathfrak{g}}^\check{}}(\lambda')\otimes V' \mapsto \mathcal{J}_{\lambda'} * \hat{Z}_{V'} * \hat{\Xi}$, we have constructed an isomorphism between the two Hom spaces. A routine diagram chase shows that isomorphism coincides with the map induced by Φ. □

7. Extending an equivalence from the subcategory of perfect complexes

7.1. **A criterion for representability.** Let algebraic stack X be given by $X = Z/G$ where Z is a quasiprojective scheme over an algebraically closed field of characteristic zero and G is a reductive group. [The results of this section are likely valid in greater generality but we present the setting needed for our applications]. We fix a G-equivariant ample line bundle L on X, such a bundle exists by Sumihiro embedding Theorem (though in examples considered in this paper X comes equipped with a supply of such line bundles).

Set $D = D^b(Coh(X))$ and let $D_{perf}(X) \subset D$ be the subcategory of perfect complexes. Set $D^{\leq n}_{perf} = D^{\leq n}(Coh(X)) \cap D_{perf}(X)$, and let $D^{\geq n}_{perf} \subset D_{perf}(X)$ be the full subcategory of objects represented by complexes of locally free sheaves placed in degree n and higher, and their direct summands.

Remark 31. It is obvious that $D^{\geq n}_{perf} \subset D^{\geq n}(Coh(X)) \cap D_{perf}(X)$. Using [25, Theorem 3.2.6] ("finiteness of finitistic dimension") one can also show that $D^{\geq n}(X) \subset D^{\geq n-\dim(Z)} \cap D_{perf}$. This implies that most of the statements below hold with $D^{\geq n}_{perf}$ replaced by $D^{\geq n}(Coh(X)) \cap D_{perf}(X)$. We neither prove nor use this point.

Proposition 32. *a) The natural functor from $D^b(Coh(X))$ to the category of contravariant functors on $D_{perf}(X)$ is fully faithful.*

b) A cohomological functor F on $D_{perf}(X)$ is represented by an object of $D(X)$ if and only if the following conditions hold:

i) For any n the functor $F|_{D^{\geq n}_{perf}}$ is represented by an object of $D_{perf}(X)$ (not necessarily by an object of $D^{\geq n}_{perf}$).

ii) There exists m such that $F|_{D^{\leq m}_{perf}} = 0$.

Proof. Fix $\mathcal{F}, \mathcal{G} \in D^b(Coh(X))$ and let $\phi_\mathcal{F}$, $\phi_\mathcal{G}$ be the corresponding functors on $D_{perf}(X)$. Fix a bounded above complex $\mathcal{F}^\bullet$ of locally free sheaves representing $\mathcal{F}$. Let $\mathcal{F}_{\geq -n} = \tau^{\text{bete}}_{\geq -n}(\mathcal{F}^\bullet)$ denote the stupid truncation.

Given a natural transformation $\phi_\mathcal{F} \to \phi_\mathcal{G}$ we get morphisms $\mathcal{F}_{\geq -n} \to \mathcal{G}$, compatible with the arrows $\mathcal{F}_{\geq -n} \to \mathcal{F}_{\geq -(n+1)}$. Choose n such that $\mathcal{F} \in D^{>-n}(Coh(X))$. Then for $N > n$ we have a canonical isomorphism $\mathcal{F} \cong \tau_{\geq -n}(\mathcal{F}_{\geq -N})$. Assuming also that $\mathcal{G} \in D^{>-n}(Coh(X))$, we get an arrow $\mathcal{F} = \tau_{\geq -n}(\mathcal{F}_{\geq -N}) \to \tau_{\geq -n}(\mathcal{G}) = \mathcal{G}$. A standard argument shows that bounded above complexes representing a given $\mathcal{F} \in D^b(Coh(X))$ form a filtered category (i.e. given two such complexes $\mathcal{F}_1^\bullet$, $\mathcal{F}_2^\bullet$, there exists a complex $\mathcal{F}_0^\bullet$ with maps of complexes $\mathcal{F}_0^\bullet \to \mathcal{F}_1^\bullet$, $\mathcal{F}_0^\bullet \to \mathcal{F}_2^\bullet$ inducing identity maps in the derived category). This implies that the arrow $\mathcal{F} \to \mathcal{G}$ does not depend on the choice of $\mathcal{F}^\bullet$.

Thus we have constructed a map $Hom(\phi_\mathcal{F}, \phi_\mathcal{G}) \to Hom(\mathcal{F}, \mathcal{G})$. It is clear from the construction that the composition $Hom(\mathcal{F}, \mathcal{G}) \to Hom(\phi_\mathcal{F}, \phi_\mathcal{G}) \to Hom(\mathcal{F}, \mathcal{G})$ is the identity map. It remains to see that the map $Hom(\phi_\mathcal{F}, \phi_\mathcal{G}) \to Hom(\mathcal{F}, \mathcal{G})$ is injective.

Let $h \in Hom(\phi_\mathcal{F}, \phi_\mathcal{G})$ be a nonzero element. Thus for some $\mathcal{P} \in D_{perf}(Coh(X))$ and $\varphi : \mathcal{P} \to \mathcal{F}$ we have $0 \neq h(\varphi) : \mathcal{P} \to \mathcal{G}$. Fix again a complex $\mathcal{F}^\bullet$, for N as above we get a distinguished triangle

$$\mathcal{F}_N[N] \to \mathcal{F}_{\geq -N} \to \mathcal{F} \to \mathcal{F}_N[N+1] \tag{12}$$

for some $\mathcal{F}_N \in Coh(X)$. For large N we have $Hom(\mathcal{P}, \mathcal{F}_N[N+1]) = 0$, thus φ factors through an arrow $\mathcal{P} \to \mathcal{F}_{\geq -N}$. It follows that for $N \gg 0$ applying h to the tautological map $\mathcal{F}_{\geq -N} \to \mathcal{F}$ we get a nonzero arrow $\mathcal{F}_{\geq -N} \to \mathcal{G}$. Since $Hom(\mathcal{F}_N[N], \mathcal{G}) = 0 = Hom(\mathcal{F}_N[N+1], \mathcal{G})$ for large N, we see that the induced arrow $\mathcal{F} \to \mathcal{G}$ is nonzero. This proves (a).

We now prove (b). We first check the "only if" direction. Condition (ii) is clear, and to check condition (i) let $\mathcal{F}$ be the representing object, and choose a bounded above complex $\mathcal{F}^\bullet$ representing $\mathcal{F}$. Setting again $\mathcal{F}_{\geq N} = \tau^{\text{bete}}_{\geq N}(\mathcal{F}^\bullet) \in D_{perf}(Coh(X))$, we claim that $Hom(\mathcal{G}, \mathcal{F}) \widetilde{\longrightarrow} Hom(\mathcal{G}, \mathcal{F}_{\geq N})$ when $\mathcal{G} \in D^{\geq m}_{perf}$, $N < m - d$, where $d = \dim(Z)$. This follows from the fact that $Ext^i(\mathcal{E}, \mathcal{K}) = 0$ for $i > d$, where $\mathcal{E}$, $\mathcal{K} \in Coh(X)$ and $\mathcal{E}$ is locally free.

To check the "if" direction, given a functor F satisfying the conditions take n in (i) satisfying $n < m - d$ where m is as in (ii) and $d = \dim(Z)$. Let $\mathcal{F}' \in D_{perf}$ be a representing object for $F|_{D^{\geq n}_{perf}}$. We claim that $\mathcal{F} = \tau_{\geq n}(\mathcal{F}')$ represents F.

First observe that

$$\mathcal{F} \in D^{>m}(Coh(X)), \tag{13}$$

to check this we need to see that $H^i(\mathcal{F}') = 0$ for $i = n, \dots, m$. If $H^i(\mathcal{F}') \neq 0$ for such an i, we can find a locally free sheaf $\mathcal{E}$ such that $Hom(\mathcal{E}, H^i(\mathcal{F}')) \neq 0$ and $Ext^{>0}(\mathcal{E}, H^j(\mathcal{F}')) = 0$ for all j (in fact, we can take $\mathcal{E} = L^{\otimes N} \otimes V$ where L is an anti-ample G-equivariant line bundle on Z and V is a representation of G). Then we get $Hom(\mathcal{E}[-i], \mathcal{F}') = F(\mathcal{E}[-i]) = 0$, which contradicts (ii).

We now construct a functorial isomorphism $F(\mathcal{G}) \cong Hom(\mathcal{G}, \mathcal{F})$, $\mathcal{G} \in D_{perf}(X)$. Fix such $\mathcal{G}$, and fix a finite complex $\mathcal{G}^\bullet$ of locally free sheaves representing $\mathcal{G}$. The desired isomorphism is obtained as the following composition:

$$Hom(\mathcal{G}, \mathcal{F}) \cong Hom(\tau^{\text{bete}}_{\geq m}(\mathcal{G}^\bullet), \mathcal{F}) \cong Hom(\tau^{\text{bete}}_{\geq m}(\mathcal{G}^\bullet), \mathcal{F}') \cong F(\tau^{\text{bete}}_{\geq m}(\mathcal{G}^\bullet)) \cong F(\mathcal{G}).$$

Here the first isomorphism follows from (13), which implies that $Hom(\tau^{\text{bete}}_{<m}(\mathcal{G}^\bullet), \mathcal{F}) = 0 = Hom(\tau^{\text{bete}}_{<m}(\mathcal{G}^\bullet)[-1], \mathcal{F})$.

The second isomorphism follows from the distinguished triangle $\tau_{<n}(\mathcal{F}') \to \mathcal{F}' \to \mathcal{F} \to \tau_{<n}(\mathcal{F}')[1]$ and the fact that $Hom(D^{\geq m}_{perf}(Coh(X)), D^{\leq n}(Coh(X))) = 0$, since $m - n > d$ and $Ext^i(\mathcal{E}, \mathcal{K}) = 0$ for $i > d$, where $\mathcal{E}, \mathcal{K} \in Coh(X)$ and $\mathcal{E}$ is locally free.

The third isomorphism is the assumption on $\mathcal{F}'$, and the last isomorphism follows from (ii). It is easy to see that the constructed isomorphism is independent on the auxiliary choices and is functorial. $\square$

Let $X = Z/G$ be as in the previous Proposition. We assume that X admits a projective G-equivariant morphism $X \to Y$ where Y is affine. Let L be a G-equivariant ample line bundle on X. We have the homogeneous coordinate ring $\hat{\mathcal{O}}(X) = \bigoplus\limits_{n \geq 0} \Gamma(L^{\otimes n})$. The assumptions on X imply that $\hat{\mathcal{O}}(X)$ is Noetherian.

We now assume that $\mathcal{C}$ is a triangulated category with a fixed full triangulated embedding $i : D_{perf}(X) \to \mathcal{C}$.

For $M \in \mathcal{C}$ we can form a module for the homogeneous coordinate ring $\tilde{\Psi}(M) = \oplus_{n \geq 0} Hom^G_{deeq}(\Phi(L^{\otimes -n}), M)$.

We also set $\tilde{\Psi}_i(M) = \oplus_{n \geq i} Hom^G_{deeq}(\Phi(L^{\otimes -n}), M)$.

Proposition 33. *For $M \in \mathcal{C}$ the following are equivalent.*

a) For any m the functor on $D^{\geq m}_{perf}(X)$, $\mathcal{F} \mapsto Hom(i(\mathcal{F}), M)$ is represented by an object of $D_{perf}(X)$.

b) The module $\tilde{\Psi}(M[n])$ is finitely generated for all n and $\tilde{\Psi}(M[n]) = 0$ for $n \gg 0$.

c) We have $\tilde{\Psi}(M[n]) = 0$ for $n \gg 0$ and for any n there exists m, such that $\tilde{\Psi}_m(M[n])$ is finitely generated.

The proof of the Proposition is based on the following

Lemma 34. *If $\tilde{\Psi}(M[n]) = 0$ for $n \geq s$, then $Hom(i(\mathcal{F}), M) = 0$ for $\mathcal{F} \in D^{>s+d+1}_{perf}$, $d = \dim(Z)$.*

Proof. We claim that any object in $\mathcal{F} \in D^{>s+d}_{perf}$ is isomorphic to a direct summand in an object represented by a complex placed in degree s and higher, with each term isomorphic to $L^{\otimes i} \otimes V$, $i \leq 0$, $V \in Rep(G)$. This clearly implies the Lemma.

It remains to check that claim. Let $\mathcal{F} \in D^{>s+d}_{perf}$. By a standard argument there exists a bounded above complex $\mathcal{F}^\bullet$ representing $\mathcal{F}$ whose terms are of the form $L^{\otimes n} \otimes V$, $n \leq 0$. Then using the fact that Ext^i from a locally free sheaf to any sheaf vanishes for $i > d$, we conclude the argument by a standard trick: consider the distinguished triangle $\mathcal{F}_s[s] \to \tau^{\text{bete}}_{\geq s}(\mathcal{F}^\bullet) \to \mathcal{F}$ and use that $Hom(\mathcal{F}, \mathcal{F}_s[s+1]) = 0$. $\square$

Proof of the Proposition. (a) clearly implies (b), while (b) implies (c). We proceed to prove that (c) implies (a).

Assume that (c) holds. In view of the Lemma, it suffices to find for every m an object $F_{M,m} \in D_{perf}(X)$ and a morphism $c_m : i(\mathcal{F}_{M,m}) \to M$ so that $\tilde{\Psi}(Cone(c_m)[l]) = 0$ for $l \geq m$. Moreover, it suffices to do so after possibly replacing the full embedding i by the functor $i' : \mathcal{F} \mapsto i \circ (\mathcal{F} \otimes L^{\otimes p})$ for some $p \in \mathbb{Z}$ (notice that conclusion of Lemma 34 is not affected by such a substitution).

Let d_0 be the largest integer such that $\tilde{\Psi}(M[d_0]) \neq 0$. We argue by descending induction in d_0. Using the finite generation condition we find a locally free sheaf $\mathcal{E} \in Coh(X)$ and a morphism $\mathcal{E}[-d_0] \to M$, such that the induced map $\tilde{\Psi}_m(\mathcal{E}) \to \tilde{\Psi}_m(M[d_0])$ is surjective for some $m \in \mathbb{Z}$. Fix $m_0 \geq 0$ such that $R^{>0}\Gamma(L^{\otimes i} \otimes \mathcal{E}) = 0$ for $i \geq m_0$. We can assume without loss of generality that $m_0 \geq m$. Then upon replacing the embedding i by $i' : \mathcal{F} \mapsto i(\mathcal{F} \otimes L^{\otimes -m_0})$ we get that $M' := Cone\,(\mathcal{E} \to M)$ satisfies: $\tilde{\Psi}(M'[i]) = 0$ for $i \geq d_0$. Also it is clear that the finite generation condition is satisfied for M', i'. Thus we can assume that the statement is true for M' by the induction assumption. Then the statement about M follows from the octahedron axiom. □

7.2. **A characterization of $D^b(Coh(Z))$ as an ambient category of $D^b_{perf}(Z)$.** We continue to assume that the functor Φ is fully faithful. Assume also that equivalent conditions of Proposition 33 hold, thus the condition of Proposition 32(b)(i) is satisfied. Assume also that assumption (b,ii) holds. In view of Proposition 32 we get a functor $\Psi : \mathcal{C} \to D^b(Coh(X))$ sending $M \in \mathcal{C}$ to $\mathcal{F} \in D^b(Coh(X))$ representing the functor $\mathcal{G} \mapsto Hom(i(\mathcal{G}), M)$ on $D_{perf}(Coh(X))$.

It is not hard to see that Ψ is a triangulated functor.

We now assume that $\mathcal{C}$ is equipped with a bounded t-structure τ. Consider the following properties of the functor Ψ in relation to the t-structures.

A) The functor Ψ is of bounded amplitude, i.e. there exists δ such that $\Psi : D^{\tau,\leq 0} \to D^{<\delta}(Coh(Z))$, $\Psi : D^{\tau,\geq 0} \to D^{>-\delta}(Coh(Z))$.

B) There exists $d \in \mathbb{Z}$ such that for $\mathcal{F} \in \mathcal{C}$ we have: $\Psi(\mathcal{F}) \in D^{\leq 0}(Coh(Z)) \Rightarrow \mathcal{F} \in \mathcal{C}^{\tau,\leq d}$.

Proposition 35. *a) Property (B) implies that Ψ is fully faithful.*

b) Properties (A), (B) imply that Ψ is an equivalence.

Proof. To unburden notation we assume without loss of generality that $d = 0$, this can be achieved by replacing the t-structure τ with its shift by $-d$.

Recall that Φ is assumed to be a full embedding. It follows that $\Psi \circ \Phi \cong Id_{D_{perf}(Z)}$. It follows from the definition of Ψ that for $\mathcal{F} \in D_{perf}(Z)$ we have:

$$Hom(\Phi(\mathcal{F}), M) \cong Hom(\mathcal{F}, \Psi(M)) \cong Hom(\Psi\Phi(\mathcal{F}), \Psi(M)).$$

Thus the map $Hom(M_1, M_2) \to Hom(\Psi(M_1), \Psi(M_2))$ is an isomorphism when $M_1 \in Im(\Phi)$.

Fix $M_1, M_2 \in \mathcal{C}$. Fix n such that $M_2 \in \mathcal{C}^{\tau,>n}$ and $\Psi(M_2) \in D^{>n}(Coh(Z))$. Fix a bounded above complex $\mathcal{F}^\bullet$ of locally free sheaves representing $\Psi(M_1)$, and let $\mathcal{F}_{\geq N} \in D_{perf}(X)$ be the naive truncation as above. We have an exact triangle $\mathcal{F}_N[-N] \to \mathcal{F}_{\geq N} \to \mathcal{F}$ for some $\mathcal{F} \in Coh(Z)$.

Assuming $N < n$, we get

$$Hom(\Psi(M_1), \Psi(M_2)) \cong Hom(\mathcal{F}_{\geq N}, \Psi(M_2)) \cong Hom(\Phi(\mathcal{F}_{\geq N}), M_2).$$

We have a morphism $\Phi(\mathcal{F}_{\geq N}) \to M_1$ whose cone lies in $D^{\tau,\leq N+1}$ in view of condition (B). [Notice that Ψ sends this cone to $\mathcal{F}_N[-N+1]$.] Thus $Hom(M_1, M_2) \cong Hom(\Phi(\mathcal{F}_{\geq N}), M_2)$, so composing the above isomorphisms we get that $Hom(M_1, M_2) \cong Hom(\Psi(M_1), \Psi(M_2))$. It is easy to see that this map coincides with the map induced by Ψ, so (a) is proved.

b) In view of (a) it remains to show essential surjectivity of Ψ. Fix $\mathcal{F} \in D^b(Coh(Z))$ and a bounded above complex of locally free sheaves $\mathcal{F}^\bullet$ representing $\mathcal{F}$. Let n be such that $\mathcal{F} \in D^{\geq n}(Coh(Z))$.

Set $N = n - 2\delta$. Set $M = \tau_{\tau, \geq n-\delta}(\Phi(\mathcal{F}^{\bullet}_{\geq N}))$. Then $\Psi(M) \in D^{>N}(Coh(Z))$ and the cone of the arrow $\Psi(M) \to \mathcal{F}_{\geq N} \cong \Psi(\Phi(\mathcal{F}_{\geq N}))$ lies in $D^{<n}$. It follows that $\Psi(M) \cong \mathcal{F}$. □

8. Compatibility between the t-structures and construction of the functor from constructible to coherent category

8.1. **Almost right exactness of Φ_{perf}.**

Proposition 36. *If $\mathcal{F} \in D$ is such that $Hom^{\leq 0}(\mathcal{F}, \Phi_{perf}(\mathcal{G})) = 0$ for $\mathcal{G} \in Coh^{G\check{}}_{perf}(St')$ then $\mathcal{F} \in D^{\leq \dim \mathfrak{g}}(\mathcal{P})$.*

Remark 37. It is not hard to show a slightly stronger statement: in fact $\mathcal{F} \in D^{\leq \dim \mathfrak{g} - \mathrm{rank}(\mathfrak{g})}$.

The proof of Proposition is preceded by some auxiliary results.

Lemma 38. *For $X \in D$ there exists a finite subset $S \subset W$, such that for large $\lambda \in \Lambda^+$ we have*

$$j_w^*(\mathcal{J}_{-\lambda} * X) \neq 0 \Rightarrow w \in (-\lambda) \cdot S \subset W; \\ j_w^*(X * J_{-\lambda}) \neq 0 \Rightarrow w \in S \cdot (-\lambda) \subset W. \tag{14}$$

Proof. By the $*$-support of an object $X \in D^b(P)$ we mean the set of points $i_x : \{x\} \hookrightarrow \mathcal{F}\ell$ such that $i_x^*(X) \neq 0$. Proper base change shows that the $*$-support of $\mathcal{J}_{-\lambda} * X$ lies in the convolution of sets $\mathcal{F}\ell_{-\lambda}$ and $Supp(X)$. This implies the first part of (14); the second one is similar. □

Lemma 39. *Let $\mathcal{F}$ be as in Proposition 36.*

For large λ and $n < -\dim \mathfrak{g}$ we have

$$Ext^n(\mathcal{J}_{-\lambda} * \mathcal{F}, j_{w*}) = 0 \tag{15}$$

for all w.

Proof. According to Lemma 38 there exists a finite set $S \subset W$ such that for large λ the left hand side of (15) vanishes for all n unless $w \in S \cdot (-\lambda)$. Also for large λ we have $S \cdot (-\lambda) \subset W_f \cdot (-\Lambda^+)$ and each element in this set is the maximal length representative of its right W_f coset. Hence for all $w \in W$ we have

$$Ext^p(\mathcal{F} * J_{-\lambda}, j_{w*}) \cong Ext^p(\mathcal{F} * J_{-\lambda}, \nabla_w * \Xi), \tag{16}$$

this follows from the fact that $\nabla_w * \Xi$ admits a filtration with associated graded $\bigoplus_{w_f \in W_f} j_{ww_f *}$, and for $w_f \neq e$ we have $Ext^\bullet(\mathcal{F} * J_{-\lambda}, j_{ww_f*}) = 0$.

Comparing Lemma 30(c) with the third diagram in Lemma 28 we get

$$\nabla_w * \Xi \cong \pi_* \circ Av_{I^0}^{right}(\nabla_w^{IW}) \cong \Phi_{perf}((pr'_{Spr,1})^*(\mathcal{G}))$$

for $\mathcal{G} = (\Phi_{IW}^{I^0})^{-1}(\nabla_w^{IW}) \in Coh^{G\check{}}(\widehat{\tilde{\mathfrak{g}}\check{}})$.

The sheaf $\mathcal{G}$ has a left resolution by locally free $G\check{}$ equivariant sheaves of length at most $\dim \mathfrak{g} = \dim(\tilde{\mathfrak{g}}\check{})$. Thus $(p'_{Spr,1})^*(\mathcal{G})$ is represented by complex of locally free sheaves in degrees $\dim \mathfrak{g}$ and higher, so the condition of Propostion 36 implies vanishing of the right hand side of (16) for $n < \dim \mathfrak{g}$. □

Proof. of Proposition 36. Lemma 39 implies that for large λ the object $\mathcal{F} * J_{-\lambda}$ lies in $D^{\leq \dim \mathfrak{g}}(\mathcal{P})$. We have

$$\mathcal{F} = (\mathcal{F} * J_\lambda) * J_{-\lambda}.$$

The functor of convolution with J_λ is right exact, since it can be rewritten as a $*$ direct image under an affine map which is right exact [4]. This shows that $\mathcal{F} \in D^{\leq \dim \mathfrak{g}}(\mathcal{P})$ as well. □

8.2. **The functor from constructible to coherent category.** Recall the functor $\tilde{\Psi}$ from D to modules over the homogeneous coordinate ring.

Proposition 40. *For $\mathcal{F} \in Perv_N(G/B) \subset \mathcal{P}$ we have $\tilde{\Psi}(\mathcal{F} * \mathcal{J}_\rho[n]) = 0$ for $n \neq 0$.*

Proof. It suffices to check that for $\mathcal{F} = j_{w!}$, j_{w*}, $w \in W_f$ we have $\tilde{\Psi}(\mathcal{F} * \mathcal{J}_\rho)[n] = 0$ for $n \neq 0$. This reduces to showing that for dominant λ, μ, ν with μ strictly dominant we have $Ext^i(\mathcal{J}_{-\lambda} * \Xi * Z_\nu * \mathcal{J}_{-\mu}, \mathcal{F}) = 0$ for $i \neq 0$. We have $\ell(w\mu) = \ell(\mu) - \ell(w)$, $\ell(\lambda w) = \ell(\lambda) + \ell(w)$ for $w \in W_f$. Thus for such w we have

$$Ext^i(\mathcal{J}_{-\lambda} * \Xi * Z_\nu * \mathcal{J}_{-\mu}, j_{w!}) = Ext^i(\mathcal{J}_{-\lambda} * \Xi * Z_\nu, j_{w!} j_{\mu*}) = Ext^i(\mathcal{J}_{-\lambda} * \Xi * Z_\nu, j_{w\mu*}),$$

$$Ext^i(\mathcal{J}_{-\lambda} * \Xi * Z_\nu * \mathcal{J}_{-\mu}, j_{w*}) = Ext^i(\Xi * Z_\nu * \mathcal{J}_{-\mu}, j_{\lambda*} j_{w*}) = Ext^i(\Xi * Z_\nu * \mathcal{J}_{-\mu}, j_{\lambda w*}).$$

Since $\Xi * \mathcal{Z}_\nu$ is titling, $\mathcal{J}_{-\lambda} * \Xi * Z_\nu$ admits a standard filtration, which shows that the first Ext group vanishes for $i \neq 0$. Likewise, $\Xi * Z_\nu * \mathcal{J}_{-\mu}$ admits a standard filtration which shows vanishing of the second Ext group for $i \neq 0$. □

Proposition 41. *The module $\tilde{\Psi}(\mathcal{F})$ is finitely generated for any $\mathcal{F} \in D$.*

Proof. For $\mathcal{F}$ in the image of Φ_{perf} this is clear from the fact that Φ_{perf} is a full embedding. Every irreducible object in $Perv(G/B)$ is a subquotient of Ξ. Then it follows from the previous Proposition that if $\mathcal{L}$ is such an irreducible object, $\tilde{\Psi}(\mathcal{L} * J_\rho)$ is a subquotient of $\tilde{\Psi}(\Xi * J_\rho)$, hence it is finitely generated (since the homogeneous coordinate ring of Steinberg variety is Noetherian), while $\tilde{\Psi}(\mathcal{L} * J_\rho[n]) = 0$ for $n \neq 0$. It follows that the same is true for any $\mathcal{L} \in Perv(G/B)$. Now it follows from Proposition 33 that $\tilde{\Psi}(J_\lambda * \mathcal{F} * J_\mu[n])$ is finitely generated for $\mathcal{F} \in Perv(G/B)$ and any λ, $\mu \in \Lambda$, $n \in \mathbb{Z}$; also, for a fixed $\mathcal{F}$ it vanishes for $n \gg 0$ for all λ, μ.

Such objects generated D, so the claim follows. □

Now $\Psi : D \to D^b(Coh^{G\check{}}(St'))$ is defined.

Corollary 42. *a) For $\mathcal{F} \in Perv_N(G/B) \subset \mathcal{P}$ we have $\Psi(\mathcal{F}) \in Coh^{G\check{}}(St')$.*

b) $\Psi(j_{w*}) \in Coh^{G\check{}}(St')$ for $w \in W^f$ and $\Psi(j_{w!}) \in Coh^{G\check{}}(St')$ when $w \in W_f\nu$, $\nu \in -\Lambda^+$.

Proof. a) follows from the Proposition.

b) follows from a) since $w \in W^f$ can be written as $w = w'\lambda$, $\lambda \in \Lambda^+$, $w' \in W_f$, so that $\ell(w) = \ell(w') + \ell(\lambda)$. Then we get $\Psi(j_{w*}) = \Psi(j_{w'*} * j_{\lambda*}) = \Psi(j_{w'*}) \otimes \mathcal{O}(0, \lambda)$. Similarly, if $w = w'\nu$, $\nu \in -\Lambda$, then $\ell(w) = \ell(w') + \ell(\nu)$. □

Proposition 43. *There exists δ, such that for $\mathcal{F} \in \mathcal{P}$ we have*

$$Hom_{deeq}^{G\check{} \times T\check{}^2}(\Phi_{perf}(\hat{\Xi}), \mathcal{F}[i]) = 0$$

for $i \notin [-\delta, \delta]$.

Proof. We need to check that for some $\delta \in \mathbb{Z}$ we have

$$Ext^i(\mathcal{J}_{-\lambda} * \hat{Z}_\nu * \hat{\Xi} * \mathcal{J}_{-\mu}, \mathcal{F}) = 0,$$

for $i \notin [-\delta, \delta]$, $\mathcal{F} \in \mathcal{P}$. It suffices to consider $\mathcal{F} = j_{w*}$ or $j_{w!}$, $w \in W$. Consider first $\mathcal{F} = j_{w*}$. We can write $w = w^f w_f$ where $w_f \in W_f$, $w^f \in W^f$. The Ext space in question is then isomorphic to

$$Ext^i(\mathcal{J}_{-\lambda} * \hat{Z}_\nu * \hat{\Xi} * \mathcal{J}_{-\mu} * j_{w_f^{-1}!}, j_{w^f_*}) = 0.$$

We already know that Ψ is a full embedding taking left convolution with Ξ to $p^*_{Spr,1} p_{Spr,1*}$. Thus the latter space is isomorphic to

$$Ext^i_{D^b(Coh^{G^\check{}}(St))}\left(\mathcal{O}(-\lambda, 0) \otimes V_\nu \otimes p^*_{Spr,1} p_{Spr,1*}\left(\mathcal{O}(-\mu, 0)\Psi(j_{w_f^{-1}!})\right), \Psi(j_{w^f_*})\right) = 0.$$

The functor $p_{Spr,1*}$ has homological dimension $d = \dim(G/B)$. The map Spr has finite Tor dimension equal to d, so $p^*_{Spr,1} p_{Spr,1*} * \mathcal{O}(-\mu, 0)\Psi(j_{w_f^{-1}!})$ is concentrated in homological degrees between $-d$ and d. Moreover, the image of $p^*_{Spr,1}$ is contained in $D_{perf}(Coh)$, since $\tilde{\mathfrak{g}}^\check{}$ is smooth. Using (the easy particular case of a Gorenstein variety in) "finiteness of finitistic dimension Theorem" [25] we get that $p^*_{Spr,1} p_{Spr,1*}\left(\mathcal{O}(-\mu, 0)\Psi(j_{w_f^{-1}!})\right) \in D^{>-3d}_{perf}$. The claim follows.

Set now $\mathcal{F} = j_{w!}$. We can write $w = w_1 w_f$ where $w_1 \in W_f(-\Lambda^+)$ and $w_f \in W_f$ and $\ell(w) = \ell(w_1) - \ell(w_f)$. Thus $j_{w!} = j_{w_1!} * j_{w_f*}$. The argument now proceeds as in the previous case. □

9. The equivalences

9.1. **Equivalence** (3)**.** We use the criterion of Propostion 32(b) to show that the functor $\mathcal{F} \mapsto Hom(\Phi_{perf}(\mathcal{F}), M)$ is represented by an object of $D^b(Coh^{G^\check{}}(St'))$; this object is then defined uniquely up to a unique isomorphism in view of Proposition 32(a) and we obtain a functor $\Psi' : D_{I^0 I} \to D^b(Coh^{G^\check{}}(St'))$ sending $M \in D_{I^0 I}$ to the corresponding representing object.

We need to check that conditions of Proposition 32(b) are satisfied. Condition 32(b)(i) (representability of the restriction to $D^{\geq n}_{perf}$ for all n) follows from Propositions 41 and 43 (finite generation and bounded amplitude) in view of Proposition 33. Condition 32(b)(ii) (vanishing on $D^{\leq m}_{perf}$ for $m \ll 0$) follows from Proposition 43.

Now the functor Ψ' is defined. Proposition 35(b) shows it is an equivalence in view of Propositions 36 and 43.

9.2. **Equivalence** (2)**.** We again use the criterion of Propostion 32(b) to show that the functor $\mathcal{F} \mapsto Hom(\widehat{\Phi}_{perf}(\mathcal{F}), M)$ is represented by an object of $D^b(Coh^{G^\check{}}_{\mathcal{N}}(St))$; this object is then defined uniquely up to a unique isomorphism in view of Proposition 32(a) and we obtain a functor $\Psi : D_{I^0 I^0} \to D^b(Coh^{G^\check{}}_{\mathcal{N}}(St))$ sending $M \in D_{I^0 I^0}$ to the corresponding representing object.

We need to check that conditions of Proposition 32(b) are satisfied. Condition 32(b)(i) (representability of the restriction to $D^{\geq n}_{perf}$ for all n) follows from Propositions 41 and 43 (finite generation and bounded amplitude) in view of Proposition 33. Condition 32(b)(ii) (vanishing on $D^{\leq m}_{perf}$ for $m \ll 0$) follows from Proposition 43. Since the torus log monodromy action on D_{I^0, I^0} is unipotent, this object is

set theoretically supported on the preimage of $\mathcal{N}$ in St. Thus we get the functor $\Psi : D_{I^0 I^0} \to D^b(Coh^{G^\check{}}_{\mathcal{N}}(St))$.

Lemma 44. *The following diagrams commute:*

$$\begin{array}{ccc} \hat{D} & \xrightarrow{\Psi} & D^b(Coh^{G^\check{}}(\widehat{St})) \\ \pi_* \big\downarrow & & \big\downarrow i^*_{St} \\ D_{I^0 I} & \xrightarrow{\Psi'} & D^b(Coh^{G^\check{}}(St')) \end{array}$$

$$\begin{array}{ccc} D_{I^0 I} & \xrightarrow{\Psi'} & D^b(Coh^{G^\check{}}(St')) \\ \pi^* \big\downarrow & & \big\downarrow i_{St*} \\ \hat{D} & \xrightarrow{\Psi} & D^b(Coh^{G^\check{}}(\widehat{St})) \end{array}$$

where i_{St} stands for the embedding $St' \to \widehat{St}$

Proof. Lemma 22 implies that both compositions in the first diagram are compatible with the action of $D^{G^\check{}}_{perf}(\widehat{St})$, i.e. if F_1 is the first composition and F_2 is the second one then $F_i(\mathcal{F}(X)) \cong i^*(\mathcal{F}) \otimes F_i(X)$ canonically for $\mathcal{F} \in D^{G^\check{}}_{perf}(\widehat{St})$, $X \in \hat{D}$.

In view of Proposition 36 it follows that the same isomorphism holds for $\mathcal{F} \in D^b(Coh^{G^\check{}}(\widehat{St}))$, thus commutativity of the first diagram follows from $F_1(\hat{\Xi}) \cong \mathcal{O} \cong F_2(\hat{\Xi})$.

The proof for the second diagram is similar. $\square$

We are now ready to prove that Ψ is an equivalence. Since we know that Ψ' is an equivalence and the essential image of $i_* : D^b(Coh^{G^\check{}}(St')) \to D^b(Coh^{G^\check{}}_{\mathcal{N}}(St))$ generates the target category, Lemma 44 shows that the essential image of Ψ generates the target category. Thus it suffices to check that Ψ is fully faithful. It is enough to see that

$$Hom(A, B) \xrightarrow{\Psi} Hom(\Psi(A), \Psi(B))$$

is an isomorphism when B is obtained from an object $B' \in D_{I^0 I}$ by forgetting the equivariance. This follows from the corresponding statement for Ψ' and Lemma 44. $\square$

9.3. **Equivalence** (4)**.**

9.3.1. *Passing from monodromic to equivariant category by killing monodromy.* Let X be a scheme with an action of an algebraic torus A. Let $\mathcal{P}_{mon}$ be the category of unipotently monodromic perverse sheaves on X.

We have an action of $\mathfrak{a} = Lie(A)$ on $\mathcal{P}_{mon}$ by log monodromy. Let $\mathbb{K}_{\mathfrak{a}}$ be the Koszul complex of the vector space $\mathfrak{a}$; in other words, $\mathbb{K}_{\mathfrak{a}}$ is the standard complex for homology of the abelian algebra $\mathfrak{a}$ with coefficients in the free module $U\mathfrak{a} = Sym(\mathfrak{a})$. Thus $\mathbb{K}_{\mathfrak{a}}$ is a graded commutative DG-algebra with $\mathfrak{a} \oplus \mathfrak{a}[1]$ as the space of generators and differential sending $\mathfrak{a}[1]$ to $\mathfrak{a}$ by the identity map. It is clear that $\mathbb{K}_{\mathfrak{a}}$ is quasi-isomorphic to the base field k and its degree zero part is the enveloping algebra $U\mathfrak{a}$.

We define a DG-category $\mathcal{P}_{eq}$ as the category of complexes of objects in $\mathcal{P}_{mon}$ equipped with an action of $\mathbb{K}_{\mathfrak{a}}$, such that the action of $\mathfrak{a} \subset \mathbb{K}^0_{\mathfrak{a}}$ coincides with the

log monodromy action. Let $D(\mathcal{P}_{eq}) = Ho(\mathcal{P}_{eq})/Ho_{acycl}(\mathcal{P}_{eq})$ be the quotient of the homotopy category by the subcategory of acyclic complexes.

Lemma 45. *a) We have a natural equivalence $D(\mathcal{P}_{eq}) \cong D_{cons}(X/A)$.*

b) Consider the functors $Forg : \mathcal{P}_{eq} \to Com(\mathcal{P}_{mon})$ and $Ind_{U(\mathfrak{a})}^{\mathbb{K}_\mathfrak{a}} : Com(\mathcal{P}_{mon}) \to \mathcal{P}_{eq}$, where the first one is the functor of forgetting the $\mathbb{K}_\mathfrak{a}$ action and the second one is the functor of induction from $U(\mathfrak{a})$ which acts by log monodromy to $\mathbb{K}_\mathfrak{a}$.

The induced functors on the derived categories fit into the following diagrams which commute up to a natural isomorphism:

$$\begin{tikzcd} D(\mathcal{P}_{eq}) \arrow[r, "Forg"] \arrow[d, "real_{eq}"'] & D^b(\mathcal{P}_{mon}) \arrow[d, "real"] \\ D(X/A) \arrow[r, "pr^*"] & D(X) \end{tikzcd}$$

$$\begin{tikzcd} D(\mathcal{P}_{eq}) \arrow[d, "real_{eq}"'] & D^b(\mathcal{P}_{mon}) \arrow[l, "Ind_{U(\mathfrak{a})}^{\mathbb{K}_\mathfrak{a}}[-d]"'] \arrow[d, "real"] \\ D(X/A) & D(X) \arrow[l, "pr_*"'] \end{tikzcd}$$

where pr denotes the projection $X \to X/A$, $real$ denotes Beilinson's realization functor [3] *and $d = \dim(\mathfrak{a})$.*

c) Suppose that $\mathcal{F},\ \mathcal{G} \in \mathcal{P}_{eq}$ are such that $Ext^{>0}_{D(X)}(\mathcal{F}^i, \mathcal{G}^j) = 0$ for all i, j. Then

$$Hom_{Ho(\mathcal{P}_{eq})}(\mathcal{F}, \mathcal{G}) \widetilde{\longrightarrow} Hom_{D(X/A)}(real_{eq}(\mathcal{F}), real_{eq}(\mathcal{G})).$$

Proof. a) Assume first that the action of A on X is free and the quotient $Y = X/A$ is represented by a scheme. The abelian category $Perv(Y)$ of perverse sheaves on Y admits a full embedding into the category $Perv_{mon}(X)$ of unipotently monodromic perverse sheaves on X, and the essential image of the embedding consists of sheaves with zero action of log monodromy. Thus we have a natural embedding $Com(Perv(Y)) \to \mathcal{P}_{eq}$ sending a complex of equivariant sheaves to the same complex equipped with zero action of $\mathfrak{a}$ and $\mathfrak{a}[1]$. We claim that the induced functor $D^b(Perv(Y)) \to D(\mathcal{P}_{eq})$ is an equivalence.

This claim is readily seen to be local on Y, i.e. it suffices to check it assuming that $X = A \times Y$ where A acts on the first factor by translations. In the latter case the category $Perv_{mon}(X)$ is readily identified with the tensor product of the abelian category $Perv(Y)$ and the abelian category of unipotently monodromic local systems of A, the latter is equivalent to the category of modules over the symmetric algebra $U(\mathfrak{a}) \cong Sym(\mathfrak{a})$ set-theoretically supported at zero. Thus the claim is clear in this case.

Let now X be general. Then an object of $D_A(X)$ is by definitiion (see [7]) a collection of objects in $D(\tilde{Y})$ given for every A equivariant smooth map $\tilde{X} \to X$ where the action of A on $\tilde{X}$ is free and $\tilde{Y} = \tilde{X}/A$, subject to certain compatibilities. We have the pull back functor $\mathcal{P}_{eq}(X) \to \mathcal{P}_{eq}(\tilde{X})$, composing it with the functor $\mathcal{P}_{eq}(\tilde{X}) \to D^b(Perv(\tilde{Y})) \cong D(\tilde{Y})$ we get the desired system of objects, the compatibilities are easy to see.

b) Commutativity of the first diagram is clear from the proof of (a) and commutativity of the second one follows by passing to adjoint functors (notice that in

view of self-duality of Koszul complex the functor $Ind_{U(\mathfrak{a})}^{\mathbb{K}_{\mathfrak{a}}}[-d]$ is *right* adjoint to the forgetful functor $Forg$).

c) By a standard argument the condition in (c) implies that

$$Hom_{Ho(\mathcal{P}_{mon})}(\mathcal{F},\mathcal{G}) \cong Hom(Forg(\mathcal{F}), Forg(\mathcal{G})).$$

We have adjoint pairs of functors compatible with the natural functor from the homotopy category to the derived category:

$$Ho(\mathcal{P}_{eq}) \xrightarrow{Forg} Ho(\mathcal{P}_{mon}) \xrightarrow{Ind} Ho(\mathcal{P}_{eq}),$$

$$D(\mathcal{P}_{eq}) \xrightarrow{Forg} D^b(\mathcal{P}_{mon}) \xrightarrow{Ind} D(\mathcal{P}_{eq}).$$

The composition in each case admits a filtration with associated graded $Id \otimes \Lambda(\mathfrak{a}[1])$, i.e. for $\mathcal{F} \in Ho(\mathcal{P}_{eq})$ or $\mathcal{F} \in D(\mathcal{P}_{eq})$ we have

$$Ind \circ Forg(\mathcal{F}) \in [\Lambda^d(\mathfrak{a}) \otimes \mathcal{F}[d]] * [\Lambda^{d-1}(\mathfrak{a}) \otimes \mathcal{F}[d-1]] * \cdots * [\mathfrak{a} \otimes \mathcal{F}[1]] * [\mathcal{F}],$$

where we used the notation of [4]: $X*Y$ is the set of objects Z such that there exists a distinguished triangle $X \to Z \to Y$. Since $Hom_{Ho(\mathcal{P}_{eq})}(Ind \circ Forg(\mathcal{F}), \mathcal{G}) \widetilde{\longrightarrow} Hom_{D(X)}(Ind \circ Forg(\mathcal{F}), \mathcal{G})$, it follows by induction in n that $Hom^n_{Ho(\mathcal{P}_{eq})}(\mathcal{F},\mathcal{G}) \widetilde{\longrightarrow} Hom^n_{D(X)}(\mathcal{F},\mathcal{G})$. □

Corollary 46. *Let $\hat{\mathcal{T}}_{II}$ denote the DG category whose objects are finite complexes of objects in $\hat{\mathcal{T}}$ equipped with an action of $\mathbb{K}_{\check{\mathfrak{t}}^2}$ such that the action of $\mathbb{K}^0_{\check{\mathfrak{t}}^2} = U(\check{\mathfrak{t}}^2)$ coincides with the action induced by the torus monodromy. Then the homotopy category $Ho(\hat{\mathcal{T}}_{II})$ is naturally equivalent to D_{II}.*

Proof. Lemma 45(c) yields a fully faithful functor $Ho(\hat{\mathcal{T}}_{II}) \to D_{II}$. To see that this functor is essentially surjective, notice that by Lemma 45(b) implies that the composition of the natural functors $Ho(\hat{\mathcal{T}}_{II}) \to Ho(\hat{\mathcal{T}}) \to Ho(\hat{\mathcal{T}}_{II})$ contains identity functor as a direct summand (more precisely, this composition is isomorphic to tensoring with $H^*(\check{T}^2) \in D^b(Vect)$). Thus every object of D_{II} is a direct summand in an object which belongs to the essential image of the full embedding $Ho(\hat{\mathcal{T}}_{II})$. Thus we will be done if we check that $Ho(\hat{\mathcal{T}}_{II}))$ is Karoubian (idempotent complete).

Since a direct summand of a free-monodromic tilting object is again free-monodromic tilting, the category $\hat{\mathcal{T}}_{II}$ is idempotent complete. For $T^\bullet \in \hat{\mathcal{T}}_{II}$ the space of closed endomorphisms of the complex commuting with the $\mathbb{K}(\check{\mathfrak{t}}^2)$ action is a pro-finite dimensional ring whose quotient by its pro-nilpotent radical is fintie dimensional. The subspace of endomorphisms homotopic to zero is a two-sided ideal in this ring. Now elementary algebra implies that every idempotent endomorphism of an object in $Ho(\hat{\mathcal{T}}_{II})$ lifts to an idempotent in the ring of endomorphisms of the corresponding object in $\hat{\mathcal{T}}_{II}$, this shows that $Ho(\hat{\mathcal{T}}_{II})$ is idempotent complete. □

We are now ready to establish (4).

Consider the category of finite complexes of objects in $Coh^{\check{G}}(\widehat{St})$ equipped with an action of $\mathbb{K}_{\mathfrak{t}^2}$ extending the action of $\mathfrak{t}^2 = (\check{\mathfrak{t}}^*)^2$ coming from the action of linear functions on $\check{\mathfrak{t}}^2$ pull-backed under the natural map $St \to \check{\mathfrak{t}}^2$. Let $Coh^{\check{G}}_{\mathbb{K}_{\mathfrak{t}^2}}(St)$ denote this category and $Ho(Coh^{\check{G}}_{\mathbb{K}_{\mathfrak{t}^2}}(St))$ be the corresponding homotopy category.

It follows from the definition of the derived coherent category of a DG-scheme (see e.g. [?]) that there exists a natural functor

$$\rho : Ho(Coh^{\check{G}}_{\mathbb{K}_{\mathfrak{t}^2}}(St)) \to DGCoh^{\check{G}}(St \overset{\mathrm{L}}{\times}_{\check{\mathfrak{t}}^2} \{0\}) = DGCoh^{\check{G}}(\tilde{\mathcal{N}} \overset{\mathrm{L}}{\times}_{\check{\mathfrak{g}}} \tilde{\mathcal{N}}).$$

Moreover, given two complexes $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in Coh^{G^{\check{}}}_{\mathbb{K}_{\mathfrak{t}^2}}(St)$ such that $Ext^{>0}_{Coh^{G^{\check{}}}(\widehat{St})}(\mathcal{F}^i, \mathcal{G}^j) = 0$ we have

$$Hom_{Ho(Coh^{G^{\check{}}}_{\mathbb{K}_{\mathfrak{t}^2}}(St))}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \widetilde{\longrightarrow} Hom(\rho(\mathcal{F}^\bullet), \rho(\mathcal{G}^\bullet)).$$

Thus equivalence $\Phi_{I^0 I^0}$ and Corollary 46 yield a fully faithful functor $\Psi_{II} : D_{II} \to DGCoh^{G^{\check{}}}(\tilde{\mathcal{N}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}^{\check{}}} \tilde{\mathcal{N}})$. Lemma 45(b) implies that the following diagram commutes (up to a natural isomorphism). The essential image of Φ_{II} contains the essential image of the functor $Ind^{\mathbb{K}_{\mathfrak{t}^2}}_{\mathcal{O}(\mathfrak{t}^{\check{}2})} : D^b(Coh^{G^{\check{}}}(\widehat{St})) \to DGCoh^{G^{\check{}}}(\tilde{\mathcal{N}} \times_{\mathfrak{g}^{\check{}}} \tilde{\mathcal{N}})$, since the diagram

$$\begin{tikzcd}
\hat{D} \arrow[r, "\Psi_{I^0 I^0}"] \arrow[d, "Av^I_{I^0} \circ \pi_*"'] & D^b(Coh^{G^{\check{}}}(\widehat{St})) \arrow[d, "Ind^{\mathbb{K}_{\mathfrak{t}^2}}_{\mathcal{O}(\mathfrak{t}^{\check{}2})}"] \\
D_{II} \arrow[r, "\Psi_{II}"] & DGCoh^{G^{\check{}}}(\tilde{\mathcal{N}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}^{\check{}}} \tilde{\mathcal{N}})
\end{tikzcd}$$

is commutative by Lemma 45(b). Since every object of $DGCoh^{G^{\check{}}}(\tilde{\mathcal{N}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}^{\check{}}} \tilde{\mathcal{N}})$ is a direct summand in an object which lies in the image of $Ind^{\mathbb{K}_{\mathfrak{t}^2}}_{\mathcal{O}(\mathfrak{t}^{\check{}2})}$ and D_{II} has been shown to be idempotent complete, the functor Ψ_{II} is an equivalence. $\square$

10. Monoidal structure

10.1. A *DG*-model for convolution of coherent sheaves.

Lemma 47. *Let X, Y be two algebraic stacks and $F = F_{\mathcal{K}} : D^b(Coh_X) \to D^b(Coh(Y))$ be a functor coming from an object in $\mathcal{K} \in D^b(Coh_{X\times Y})$, i.e. $F : \mathcal{F} \mapsto pr_{2*}(\mathcal{K} \overset{\mathrm{L}}{\otimes} pr_1^*(\mathcal{F}))$. Let $M \in D^b(Coh(X))$ be represented by a complex of sheaves $M^\bullet$ such that $F(M^i) \in Coh(Y)$. Then $F(M)$ is canonically isomorphic to the object represented by $F(M^\bullet)$.*

Proof. A functor of the form described in the proposition lifts to a functor between filtered derived categories $F^{fil} : DF(Coh(X)) \to DF(Coh(Y))$. Recall that DF contains the homotopy category of complexes of objects in $Coh(X)$ as a full subcategory, the canonical functor from the filtered derived category to the derived category restricted to this subcategory coincides with the canonical functor from the category of complexes to the derived category. The conditions of the Lemma show that F^{fil} sends the object corresponding to the complex $M^\bullet$ to the object corresponding to $F(M^\bullet)$, which yields the desired statement. $\square$

Corollary 48. *Let $X \to Y$ be a semi-small morphism of smooth varieties equipped with an action of a reductive algebraic group H.*

a) Let $\mathcal{F}^\bullet, \mathcal{G}^\bullet$ be finite complexes of H equivariant coherent sheaves on $X \times_Y X$ such that the convolution $F^i * \mathcal{G}^j$ lies in $Coh(X \times_Y X)$ for all i, j. Let $\mathcal{F}$, $\mathcal{G}$ be the corresponding objects in the derived category. Then $\mathcal{F} * \mathcal{G}$ is canonically isomorphic to the object represented by the bicomplex $\mathcal{F}^i * \mathcal{G}^j$.

b) Assume that three complexes $\mathcal{F}^\bullet$, $\mathcal{G}^\bullet$, $\mathcal{K}^\bullet$ of H equivariant coherent sheaves on $X \times_Y X$ are such that $\mathcal{F}^i * \mathcal{G}^j$, $\mathcal{G}^j * \mathcal{K}^l$ and $\mathcal{F}^i * \mathcal{G}^j * \mathcal{K}^l$ lie in $Coh^H(X \times_Y X)$ for all i, j, l. Then the two isomorphisms between $\mathcal{F}^\bullet * \mathcal{G}^\bullet * \mathcal{K}^\bullet \in D^b(Coh^H(X \times_Y X)$ and the complex represented by $\mathcal{F}^i * \mathcal{G}^j * \mathcal{K}^l$ provided by part (a) coincide.

Proof. The convolution product comes from a functor

$$F : D^b(Coh^H((X \times_Y X)^2) \to D^b(Coh^H(X \times_Y X))$$

of the type considered in Lemma 47, namely we have $F = F_{\mathcal{K}}$, where $\mathcal{K} \in D^b(Coh^H(X \times_Y X)^3)$ is given by $\mathcal{K} = \upsilon^* \delta_*(\mathcal{O}_{X^3})$; here υ stands for the embedding $(X \times_Y X)^3 \to (X \times X)^3 = X^6$ and $\delta : X^3 \to X^6$ is given by $(x_1, x_2, x_3) \mapsto (x_3, x_1, x_1, x_2, x_2, x_3)$. Thus statement (a) follows from Lemma 47.

b) follows by considering the functor between filtered derived categories $DF(Coh^H(X \times_Y X)^3) \to DF(Coh^H(X \times_Y X))$ corresponding to the triple convolution. □

Lemma 49. *Let $X \to Y$ be a semi-small morphism of smooth quasi-projective varieties equipped with an action of a reductive algebraic group H. For $\mathcal{F} \in D^b(Coh^H(X \times_Y X))$ let $a(\mathcal{F})$ denote the corresponding functor $D^b(Coh^H(X)) \to D^b(Coh^H(X))$.*

For $\mathcal{F} \in Coh^H(X \times_Y X)$, $\mathcal{F}' \in D^b(Coh^H(X \times_Y X))$ any isomorphism of functors $a(\mathcal{F}) \cong a(\mathcal{F}')$ comes from a unique isomorphism $\mathcal{F} \cong \mathcal{F}'$.

Proof. An equivariant coherent sheaf $\mathcal{F}$ can be reconstructed from the corresponding module $M(\mathcal{F})$ over the homogeneous coordinate ring,

$M(\mathcal{F}) = \bigoplus\limits_{n,m \geq 0} \Gamma(F \otimes pr_1^*(L^n) \otimes pr_2^*(L^m))$, where L is an equivariant ample line bundle on X. Thus Lemma follows from the following expression for $M(\mathcal{F})$ in terms of the functor of convolution by $\mathcal{F}$: $M(\mathcal{F}) = \bigoplus\limits_{m,n} Hom_{deeq}(L^{-n}, \mathcal{F} * L^m)$ □

10.2. **Monoidal structure on $\Phi_{I^0 I^0}$.**

Lemma 50. *The equivalence $\widehat{\Phi}_{I^0 I^0}$ is compatible with the action on $\widehat{\Phi}_{IW}$ via the equivalence $\widehat{\Phi}_{IW}^{I^0}$, i.e. we have a functorial isomorphism*

$$\widehat{\Phi}_{IW}(\mathcal{F} * \mathcal{G}) \cong \widehat{\Phi}_{I^0 I^0}(F) * \widehat{\Phi}_{IW}(\mathcal{G})$$

where $\mathcal{F} \in D^b(Coh^{G\check{}}(\widehat{St}))$, $\mathcal{G} \in D^b(Coh^{G\check{}}(\tilde{\mathfrak{g}}\check{}))$.

Proof. For $\mathcal{F} \in D^{G\check{}}_{perf}(\widehat{St})$ this is Proposition 29.

Let now $\mathcal{F}$ be general. For any sufficiently large N we can find $\mathcal{F}' \in D^{G\check{}}_{perf}(\widehat{St})$ such that $\mathcal{F} = \tau_{\geq -N}(\mathcal{F}')$. The functor $D^b(Coh^{G\check{}}(\widehat{St})) \to D^(Coh^{G\check{}}(\widehat{\tilde{\mathfrak{g}}}\check{}))$, $\mathcal{F} \mapsto \mathcal{F} * \mathcal{G}$ has bounded homological amplitude; the functor $\hat{D} \to \hat{D}_{IW}$ $X \mapsto X * \widehat{\Phi}_{IW}(\mathcal{G})$ has bounded homological amplitude and by Proposition 36 the functor $\widehat{\Phi}$ has homological amplitude bounded above, i.e. it sends $D^{\leq 0}(Coh^{G\check{}}(\widehat{St}))toD^{\leq n}(\hat{\mathcal{P}})$ for some n. It follows that for $N \gg m \gg 0$ and $\mathcal{F}'$ as above we have

$$\widehat{\Phi}_{IW}(\mathcal{F}*\mathcal{G}) \cong \widehat{\Phi}_{IW}(\tau_{\geq -m}(\mathcal{F}'*\mathcal{G})) \cong \tau_{\geq -m}\Phi_{IW}(\mathcal{F}'*\mathcal{G}) \cong \tau_{\geq -m}(\widehat{\Phi}(\mathcal{F})*\widehat{\Phi}_{IW}(\mathcal{G})) \cong \widehat{\Phi}(\mathcal{F}) * \widehat{\Phi}_{IW}(\mathcal{G}),$$

which proves the Lemma. □

We are now ready to equip $\Phi_{I^0 I^0}$ with a monoidal structure. We work with the inverse equivalence $\Psi_{I^0 I^0}$. We need to construct an isomorphism

$$\Psi_{I^0 I^0}(\mathcal{F} * \mathcal{G}) \cong \Psi_{I^0 I^0}(\mathcal{F}) * \Psi_{I^0 I^0}(\mathcal{G}) \tag{17}$$

compatible with the associativity isomorphism.

By Lemma 50 we have an isomorphism

$$a(\Psi_{I^0I^0}(\mathcal{F}*\mathcal{G}))\cong a(\Psi_{I^0I^0}(\mathcal{F})*\Psi_{I^0I^0}(\mathcal{G}))$$

(notations of Lemma 49), which is compatible with the associativity isomorphim. Since $\Psi_{I^0I^0}:\hat{\mathcal{T}}\to Coh^{G\check{}}(\widehat{St})$, Lemma 49 yields (17) in the case when $\mathcal{F},\mathcal{G}\in\hat{\mathcal{T}}$, which is compatible with the associativity isomorphim for three objects in $\hat{\mathcal{T}}$. Now Corollary 48 yields (17) in general and shows it is compatible with associativity.

10.3. **Monoidal structure on Φ_{II}.**

10.3.1. *A monoidal structure on $Ho(\hat{\mathcal{T}}_{II})$.* In order to equip Φ_{II} with a monoidal structure we describe the monoidal structure on D_{II} in terms of the DG-model $\hat{\mathcal{T}}_{II}$ (see Corollary 46).

Let $\hat{\mathcal{T}}_{II}^{(2)}$ denote the category of finite complexes of objects in $\hat{\mathcal{T}}$ equipped with an action of $\mathcal{K}_{\mathfrak{t}^2}\otimes\Lambda(\mathfrak{t}[1])$, and $\hat{\mathcal{T}}_{II}^{(3)}$ be the category of finite complexes of objects in $\hat{\mathcal{T}}$ equipped with an action of $\mathcal{K}_{\mathfrak{t}^2}\otimes\Lambda(\mathfrak{t}^2[1])$. In both cases we require that $\mathfrak{t}^2\subset\mathcal{K}_{\mathfrak{t}^2}$ acts by logarithm of monodromy.

We have a functor $\hat{\mathcal{T}}_{II}\times\hat{\mathcal{T}}_{II}\xrightarrow{\star}\hat{\mathcal{T}}_{II}^{(2)}$ sending (T_1,T_2) to the convolution T_1*T_2; the latter complex is equipped with two actions of $\mathcal{K}_{\mathfrak{t}}$ coming respectively from the left action on T_1 and the right action on T_2. To define the action of $\Lambda(\mathfrak{t}[1])$ observe that the right monodromy action on T_1 and the left monodromy action on T_2 induce the same action on T_1*T_2, the diagonal action of $\mathcal{K}_{\mathfrak{t}}$ kills the augmentation ideal of $\mathcal{K}_{\mathfrak{t}}^0=Sym(\mathfrak{t})$, thus it factors through an action of $\Lambda(\mathfrak{t})$.

Similarly, we have a functor $\hat{\mathcal{T}}_{II}\times\hat{\mathcal{T}}_{II}\times\hat{\mathcal{T}}_{II}\xrightarrow{\star_2}\hat{\mathcal{T}}_{II}^{(3)}$ sending (T_1,T_2,T_3) to $T_1*T_2*T_3$ where the two actions of $\mathcal{K}_{\mathfrak{t}}$ come respectively from the left action on T_1 and the right action on T_3, and the two actions of $\Lambda(\mathfrak{t}[1])$ come from the diagonal action of $\mathcal{K}$ on the first and the second factor, and the diagonal action of $\mathcal{K}$ on the second and the third factor respectively. We use the same notation $\star$, $\star_2$ for the corresponding functors on the homotopy categories.

Furthermore, we have functors $\mu:Ho(\hat{\mathcal{T}}_{II}^{(2)})\to Ho(\hat{\mathcal{T}}_{II})$, $\mu:M\mapsto\mathcal{M}\overset{\mathrm{L}}{\otimes}_{\Lambda(\mathfrak{t}[1])}k$ and $\mu^{(2)}:Ho(\hat{\mathcal{T}}_{II}^{(3)})\to Ho(\hat{\mathcal{T}}_{II})$, $\mu^{(2)}:M\mapsto M\overset{\mathrm{L}}{\otimes}_{\Lambda(\mathfrak{t}^2[1])}k$.

The following Proposition obviously yields a monoidal structure on the equivalence (4).

Proposition 51. *a) The product $(M_1,M_2)\mapsto\mu(M_1\star M_2)$ makes $Ho(\hat{\mathcal{T}}_{II})$ into a monoidal category, where the associativity constraint comes from the natural isomorphisms*

$$(M_1\otimes M_2)\otimes M_3\cong\mu^{(2)}\star_2(M_1,M_2,M_3)\cong M_1\otimes(M_2\otimes M_3). \tag{18}$$

b) The equivalence $real_{eq}:Ho(\hat{\mathcal{T}}_{II})\cong D_{II}$ is naturally enhanced to a monoidal functor.

c) The equivalence $Ho(\hat{\mathcal{T}}_{II})\cong DGCoh^{G\check{}}(\tilde{\mathcal{N}}\overset{\mathrm{L}}{\times}_{\mathfrak{g}\check{}}\tilde{\mathcal{N}})$ is naturally enhanced to a monoidal functor.

Proof. To check (a) and (b) it suffices to provide a bi-functorial isomorphism

$$real_{eq}(M_1\star M_2)\cong real_{eq}(M_1)*real_{eq}(M_2)$$

sending the isomorphism (18) to the associativity constraint in D_{II}. This follows from the next Lemma.

c) follows from the definition of convolution in $DGCoh^{G\check{}}(\tilde{\mathcal{N}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}\check{}} \tilde{\mathcal{N}})$. □

To state the next Lemma, return to the setting of 9.3.1. Let X be an algebraic variety equipped with an action of an algebraic torus A and let $f : X/A \to Y$ be a map where Y is an algebraic variety and X/A is the stack quotient. Let $pr : X \to X/A$ be the projection and set $\tilde{f} = f \circ pr : X \to Y$.

Lemma 52. *a) Let $M^\bullet \in \mathcal{P}_{eq}$ be a complex of monodromic perverse sheaves on X equipped with a $\mathcal{K}_{\mathfrak{a}}$ action and let $\bar{M}$ be the corresponding object in $D_{cons}(X/A)$ (see Lemma 45).*

Assume that $\tilde{f}_*(M^i)$ is a perverse sheaf for all i.

We then have a canonical isomorphism

$$f_*(\bar{M}) \cong \tilde{f}_*(M^\bullet) \overset{\mathrm{L}}{\otimes}_{\Lambda(\mathfrak{a})} k.$$

b) Assume that a torus A' acts on X, Y so that f is A' equivariant and the action on X commutes with A.

Let $M^\bullet \in \mathcal{P}_{eq}$ be a complex of monodromic perverse sheaves on X equipped with a $\mathcal{K}_{\mathfrak{a}\oplus\mathfrak{a}'}$ action and let $\bar{M}$ be the corresponding object in $D_{cons}(X/A)$.

Assume that $\tilde{f}_*(M^i)$ is a perverse sheaf for all i.

We then have a canonical isomorphism of objects in $Ho(\mathcal{P}_{eq}(Y)) \cong D_{A'}(Y)$:

$$f_*(\bar{M}) \cong \tilde{f}_*(M^\bullet) \overset{\mathrm{L}}{\otimes}_{\Lambda(\mathfrak{a})[1]} k.$$

Proof. a) is a particular case of b), while b) follows from the following two statements:

1) The equivalence of Lemma 45(a) satisfies the following functoriality. Consider an A-equivariant map of schemes $f : X \to Y$ and use Lemma **??**(a) to identify $D_A(X) \cong D(\mathcal{P}_{eq}(X))$, $D_A(Y) \cong D(\mathcal{P}_{eq}(Y))$. Then for $\mathcal{F}^\bullet \in \mathcal{P}_{eq}(X) \cong D_A(X)$ such that $\pi_*(\mathcal{F}^i) \in Perv(Y)$ the object of $\mathcal{P}_{eq}(Y)$ obtain from $\mathcal{F}^\bullet$ by term-wise application of f_* corresponds to the object $f_*(\mathcal{F}) \in D_A(Y)$.

An argument similar to the proof of Lemma 45(c) to the particular case when the group A is trivial. This case is treated in [3].

2) For a subtorus A' of A the functor $Res^{\mathbb{K}_{\mathfrak{a}}}_{\mathbb{K}_{\mathfrak{a}'}} : \mathcal{P}^A_{eq}(X) \to \mathcal{P}^{A'}_{eq}(X)$ corresponds under the equivalence of Lemma **??**(a) to the restriction of equivariance functor $Res^A_{A'} : D_A(X) \to D_{A'}(X)$.

This is a straightforward generalization of Lemma 45(b).

Now (1) applied to the torus $A \times A'$ acting compatibly on X, Y followed by (2) applied to the subtorus A' in $A \times A'$ which acts on Y yields the Lemma. □

10.4. **Compatibility of** (3) **with the action of categories from** (2)**,** (4)**.** To finish the proof of Theorem 1 it remains to establish compatibility of equivalence (2) with the structure of a module category over the monoidal categories appearing in (2) and (4).

To check compatibility with the action of $D_{I^0I^0} \cong D^b(Coh^{G\check{}}_{\mathcal{N}}(St))$ we pass to the pro-completions and check compatibility of (3) with the action of $\hat{D} \cong D^b(Coh^{G\check{}}(\widehat{St}))$. We have an action of the monoidal category of free monodromic tilting complexes $\hat{\mathcal{T}}$ on the category of tilting objects $\mathcal{T} \subset \mathcal{P}$. which induces a structure of a module category for $Ho(\hat{\mathcal{T}})$ on $Ho(\mathcal{T})$. An argument parallel to that of section 10.2 shows that this module structure is compatible with one arising from the equivalences $Ho(\mathcal{T}) \cong D$, $Ho(\hat{\mathcal{T}}) \cong \hat{D}$, as well as with the one arising from

the equivalences $Ho(\mathcal{T}) \cong D^b(Coh^{G\check{}}(St'))$, $Ho(\hat{\mathcal{T}}) \cong D^b(Coh^{G\check{}}(\widehat{St}))$, which gives compatibility with the action of categories in (2).

To check compatibility with the action of $D_{II} \cong DGCoh^{G\check{}}(\tilde{\mathcal{N}} \overset{\mathrm{L}}{\times}_{\mathcal{N}} \tilde{\mathcal{N}})$ we use Lemma 45 to identify D_{I^0I} with the homotopy category of complexes in $\hat{\mathcal{T}}$ equipped with an action of $\mathbb{K}_{\mathfrak{t}}$ compatible with the right log monodromy action. This category of complexes carries a natural action of the monoidal DG-category of complexes in $\hat{\mathcal{T}}$ with a compatible action of $\mathbb{K}_{\mathfrak{t}^2}$. The resulting triangulated module category is module equivalent to both $D^b(Coh^{G\check{}}(St'))$ and D_{I^0I} by an argument parallel to that of section 10.3.

This establishes the compatibilities thereby completing the proof of Theorem 1.

11. Further properties

In this section we mention further properties and generalizations of the constructed equivalences.

11.1. **Exactness and Hodge D-modules.** Recall that in view of Corollary 42(a), the restriction of the functors Ψ_{I^0I}, $\Psi_{I^0I^0}$ to the subcategory of sheaves supported on the finite dimensional flag variety $G/B \subset \mathcal{F}\ell$ is t-exact, i.e. it sends a perverse sheaf to a coherent sheaf.

On the other hand, a well known result in representation theory asserts that the category O for Langlands dual Lie algebras are equivalent, i.e. we have an equivalence of abelian categories

$$\Upsilon : Perv_{U\check{}}(G\check{}/B\check{}) \widetilde{\longrightarrow} Perv_U(G/B) = Perv_{I^0}(G/B).$$

This allows to state a relation between the restriction of our equivalence Φ_{I^0I} to $Perv_{I^0}(G/B) \subset \mathcal{P}_{I^0I}$ and Hodge D-module theory.

Notice that the stack $St'/G\check{}$ can be interpreted as the *cotangent* to the stack $U\check{}\backslash G\check{}/B\check{}$. Thus for a $U\check{}$ equivariant D-module M on $G\check{}/B\check{}$ equipped with a $U\check{}$ equivariant good filtration we get $gr(M) \in Coh^LG(St')$.

Let $\mathcal{MH}_{U\check{}}(G\check{}/B\check{})$ be the category of mixed Hodge modules on $G\check{}/B\check{}$ equivariant with respect to $U\check{}$. We have forgetful functor $Forg : \mathcal{MH}_{U\check{}}(G\check{}/B\check{}) \to D{-}mod_{U\check{}}(G\check{}/B\check{}) \cong Perv_{U\check{}}(G\check{}/B\check{})$ where the second equivalence is the Riemann-Hilbert functor. Recall that a part of the data of a mixed Hodge structure on a D-module is a good filtration, i.e. for $\tilde{M} \in \mathcal{MH}_{U\check{}}(G\check{}/B\check{})$ the D-module $M = Forg(\tilde{M})$ is equipped with a canonical good filtration. Thus we get a functor $gr : \mathcal{MH}_{U\check{}}(G\check{}/B\check{}) \to Coh^{G\check{}}(St')$.

Conjecture 53. *For $M \in \mathcal{MH}_{U\check{}}(G\check{}/B\check{})$ we have a canonical isomorphism*

$$gr(M) \otimes \mathcal{O}(-\rho) \cong \Psi_{I^0I}(\Upsilon(M)).$$

This Conjecture should be compared to the results of Ben-Zvi and Nadler [6].

Example 54. Recall that the finite Weyl group W_f acts on the open subvariety $\tilde{\mathfrak{g}}\check{}^{reg} \subset \tilde{\mathfrak{g}}\check{}$.

For $w \in W_f$ let $\Gamma_w \subset St$ be the closure of the graph of w. Let Γ'_w be the scheme theoretic intersection $\Gamma_w \cap St'$. Once can show that:

$$\Psi_{I^0I} : \Xi \mapsto \mathcal{O}_{St'},$$

$$\Psi_{I^0I} : j_{w*} \mapsto \mathcal{O}_{\Gamma'_w},$$

$$\Psi_{I^0I} : j_{w!} \mapsto \Omega_{\Gamma'_w},$$

where $\Omega_{\Gamma'_w}$ is the dualizing sheaf for the Cohen-Macaulay variety Γ'_w (the Cohen-Macaulay property is proven in [11]). Parallel results for assocaited graded of Hodge D-modules will be shown in [12].

11.2. **Lusztig's cells.** In order to simplify the statement this subsection we assume that G is simply-connected, thus W_{aff} is a Coxeter group. Recall the notion of a *two sided cell* in W_{aff}. These are certain subsets in W_{aff}. In [22] Lusztig has established a bijection between 2-sided cells in W_{aff} and the set $\mathcal{N}/G\check{}$ of nilpotent conjugacy classes in $\mathfrak{g}\check{}$. The set of two sided cells is equipped with a partial order. It has been conjectured by Lusztig and proved in [8] that this order matches the adjacency order on the set of nilpotent orbits under the bijection between two-sided cells and $\mathcal{N}/G\check{}$.

The following result can be shown by an argument similar to that of [8].

Theorem 55. *Let $\underline{c}$ be a two sided cell in W_{aff} and $O_{\underline{c}} \subset \mathcal{N}$ be the corresponding nilpotent orbit.*

Let $D^{\leq \underline{c}}_{I^0 I} \subset D_{I^0 I}$ be the thick subcategory generated by irreducible objects $IC_w \in \mathcal{P}$, $w \in \underline{c}' \leq \underline{c}$.

Let $D^b(Coh^{G\check{}}_{O_{\underline{c}}}(St'))$ be the full subcategory in $D^b(Coh^{G\check{}}(St'))$ consisting of complexes whose cohomology is set-theoretically supported on the preimage of the closure of $O_{\underline{c}}$.

Then $D^b(Coh^{G\check{}}_{O_{\underline{c}}}(St'))$ the image of $D^{\leq \underline{c}}_{I^0 I}$ under the equivalence $\Psi_{I^0 I}$.

We finish by sketching some generalizations of the equivalences described in the paper. We expect they can obtained by similar methods.

11.3. **Nonunipotent monodromy.** Consider the category of $\mathbf{I}^2$ monodromic sheaves on $\widetilde{\mathcal{F}\ell}$ with a fixed eigenvalue of monodromy. The latter corresponds to a tame rang one local system on $T\check{}^2$, such local systems are in bijection with elements of $T\check{}^2$ (a subset of that in the l-adic setting in the l-adic setting). For θ_1, $\theta_2 \in T\check{}$ let D_{θ_1,θ_2} be the category of monodromic sheaves on $\widetilde{\mathcal{F}\ell}$ with corresponding eigenvalues of monodromy.

Let $\widetilde{G\check{}} \subset G\check{} \times G\check{}/B\check{}$ be the closed subvariety given by $\widetilde{G\check{}} = \{(g,x) \mid g(x) = x\}$. We have a projection $\widetilde{G\check{}} \to T\check{}$. Set $St_{grp} = \widetilde{G\check{}} \times_{G\check{}} \widetilde{G\check{}}$, and for t_1, $t_2 \in T\check{}$ let St^{t_1,t_2}_{grp} be the preimage of (t_1,t_2) under the projection $St_{grp} \to T\check{} \times T\check{}$.

Conjecture 56. *We have a canonical equivalence of triangulated categories:*

$$D_{\theta_1,\theta_2} \cong D^b\left(Coh^{G\check{}}_{St_{grp}\theta_1,\theta_2}(St_{grp})\right).$$

This Conjecture presents a generalization of (2), one can also state similar generalizations of (3), (4).

11.4. **Parabolic-Whittaker categories.** Let P be a parabolic subsgroup in G, and let $\mathbf{I}_P \subset \mathbf{G_O}$ be the parahoric subgroup which is the preimage of P under the projection $\mathbf{G_O} \to G$. Let $\mathcal{F}\ell_P = \mathbf{G_F}/\mathbf{I}_P$ be the corresponding partial affine flag variety.

Let Q be another parabolic subgroup and let ψ_Q be an additive character of $\mathbf{I}^0$ vanishing on the finite simple roots which are not in the Levi subgroup of Q as well as on the affine root and not vanishing on the simple roots in the Levi of Q. Let $D_{IW_Q}(\mathcal{F}\ell_P)$ be the corresponding category of partial Whittaker sheaves.

Let $Q^\vee$, $P^\vee$ be the corresponding parabolic subgroups in $G^\vee$. Define $\tilde{\mathfrak{g}}^\vee{}_{Q^\vee} \subset G^\vee/Q^\vee \times \mathfrak{g}^\vee$, $\tilde{\mathcal{N}}_{P^\vee} \subset G^\vee/P^\vee \times \mathfrak{g}^\vee$ by: $\tilde{\mathfrak{g}}^\vee{}_{Q^\vee} = \{(\mathfrak{q}, x) \mid x \in \mathfrak{q}\}$, $\tilde{\mathcal{N}}_{P^\vee} = \{(\mathfrak{p}, x) \mid x \in rad(\mathfrak{p})\}$, where we used the identification between $G^\vee/Q^\vee$, respectively $G^\vee/P^\vee$ and the corresponding conjugacy class of parabolic subalgebras, and rad denotes nilpotent radical.

Conjecture 57. *We have a canonical equivalence*

$$D_{IW,Q}(\mathcal{F}\ell_P) \cong D^b(Coh^{G^\vee}(\tilde{\mathfrak{g}}^\vee{}_{Q^\vee} \times_{\mathfrak{g}^\vee} \tilde{\mathcal{N}}_{P^\vee})).$$

There are natural pull-back, push-forward and Iwahori-Whittaker averaging between the categories of constructible sheaves which should correspond to the functors between the derived categories of coherent sheaves given by the natural correspondences.

Finally, let us mention the *Koszul duality* functors which give equivalences between the graded version of $D_{IW,Q}(\mathcal{F}\ell_P)$ and $D_{IW,P}(\mathcal{F}\ell_Q)$, see [13]. Under the equivalence of Conjecture 57 these should correspond to *linear Koszul duality* [23], this would provide a categorification of the main result of [24].

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